Global Optimization of H_∞ problems: Application to robust control synthesis under structural constraints

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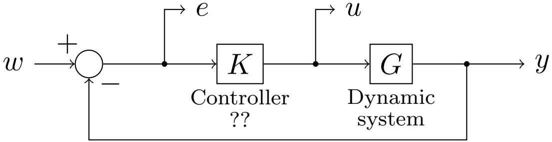
Plan

1. Frequency constraints on systems and H_∞ norm
2. Min max optimization
3. Application
4. Conclusion

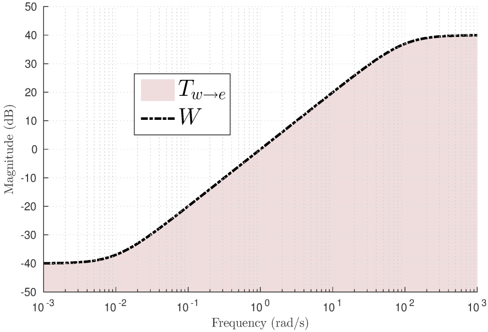
Problem statement



- The transfer function from w to e is denoted $T_{w\to e}(s)$, $e = T_{w\to e}w$.
- $T_{w\to e}(s)$ depends on K.
- Frequency constraint on e: $T_{w\to e}(i\omega) \leq W(i\omega)$.
- $||.||_\infty$ is defined by: $||T_{w\to e}||_\infty = \sup_\omega \bar{\sigma}(T_{w\to e})$.
- If the system has one input, $\sup_\omega \bar{\sigma}(T_{w\to e}) = \sup_\omega(|T_{w\to e}|)$

Example of frequency constraint

Frequency constraint on e: $T_{w\to e}(i\omega) \leq W(i\omega)$.



This constraint is respected if $||T_{w\to e}W^{-1}||_\infty \leq 1$.

Problem considered

In our example, we want to find K such as:

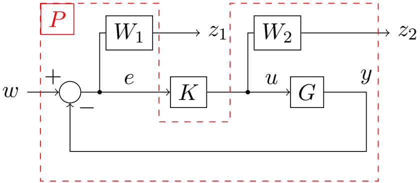
$$\begin{cases} T_{w\to e}(i\omega) \leq W_1^{-1}(i\omega) \\ T_{w\to u}(i\omega) \leq W_2^{-1}(i\omega) \\ K \text{ stabilizes the closed-loop system} \end{cases}$$

$$T_{w\to e}(i\omega) = \frac{1}{1 + G(i\omega)K(i\omega)}$$

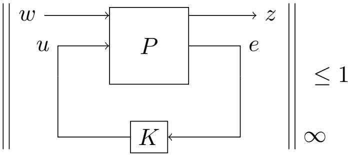
$$T_{w\to u}(i\omega) = \frac{K(i\omega)}{1 + G(i\omega)K(i\omega)}$$

H_∞ resolution

The classical way to solve this problem is to create an augmented system P



and to solve the problem



Structural constraint

- Minimization of the H_∞ norm under a stability constraint:

  $\rightarrow$ usually formulated as a semi definite convex program (minimization of a linear objective under linear matrix inequality constraint).

- However, we impose a structural constraint on the controller:

$$K(s) = k_p + \frac{k_i}{s} + \frac{k_d s}{1+s}.$$

  $\rightarrow$ The problem is no longer convex!

Min max problem formulation

As we consider a single input system, the problem can also be formulated as follow:

$$\begin{cases} \min_{k} \sup_{\omega} \max \left(|W_1(i\omega) T_{w\to e}(i\omega)|, |W_2(i\omega) T_{w\to u}(i\omega)| \right), \\ s.t. \quad K \text{ stabilizes the closed-loop system} \end{cases}$$

Stability:

- ensure with the Routh-Hurwitz criterion.
- $\rightarrow$ satisfaction of a polynomial system: $R(k_p, k_i, k_d) \leq 0$.

Plan

Min max problem formulation

- Our optimization problem is the following:

$$\begin{cases} \min_{k \in \mathbb{K}} \sup_{\omega \in [\omega]} f(k, \omega), \\ \\ s.t. \quad R(k) \leq 0 \end{cases}$$

where $f(k, \omega) = \max\left(\left|T_{w \to e}(i\omega)\right|, \left|T_{w \to u}(i\omega)\right|\right)$.

- $\mathbb{K}$ is the set of possible coefficients of $k = \begin{pmatrix} k_p \\ k_i \\ k_d \end{pmatrix}$.
- $[\omega]$ is the domain of ω.

We solve the minimization problem with a branch and bound algorithm on k.

Main Branch and bound algorithm: minimization

Interval B&B algorithm:

While $\mathcal{L} \neq \emptyset$:

1. Choose a box $\boldsymbol{k}$ from $\mathcal{L}$.
2. Contract $\boldsymbol{k}$ w.r.t $R(\boldsymbol{k}) \leq 0$.
3. Compute $[lb_{\boldsymbol{k}}, ub_{\boldsymbol{k}}]$ an enclosure of $\sup_{\omega} \mathbf{f}(\boldsymbol{k}, \omega)$.
4. Try to find a good feasible solution in $\boldsymbol{k}$.
5. Update best current solution.

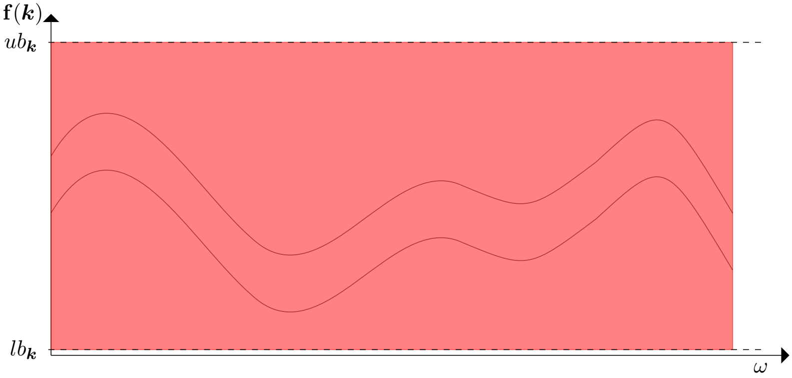
Stop criterion: $width(\boldsymbol{k}) < \epsilon$

Computing an enclosure of $\sup_\omega \mathbf{f}(\boldsymbol{k}, \omega)$

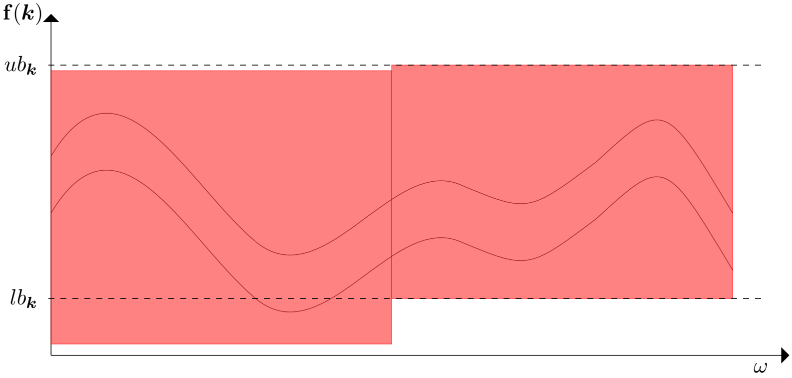
Given $\boldsymbol{k} \subseteq \mathbb{K}$, we need to compute an enclosure $[lb_{\boldsymbol{k}}, ub_{\boldsymbol{k}}]$ of $\sup_\omega \mathbf{f}(\boldsymbol{k}, \omega)$:

$\rightarrow$ We use a secondary branch and bounds algorithm.

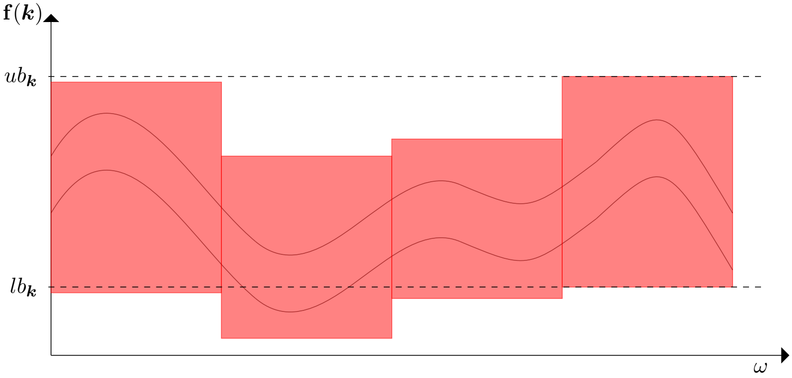
Secondary Branch and Bound algorithm: maximization



Secondary Branch and Bound algorithm: maximization



Secondary Branch and Bound algorithm: maximization



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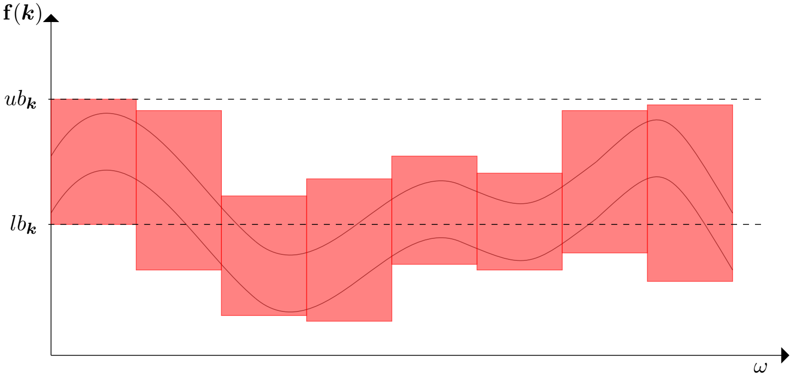
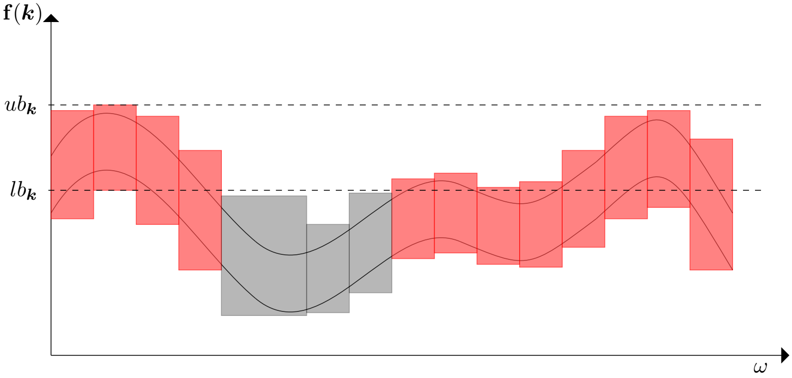
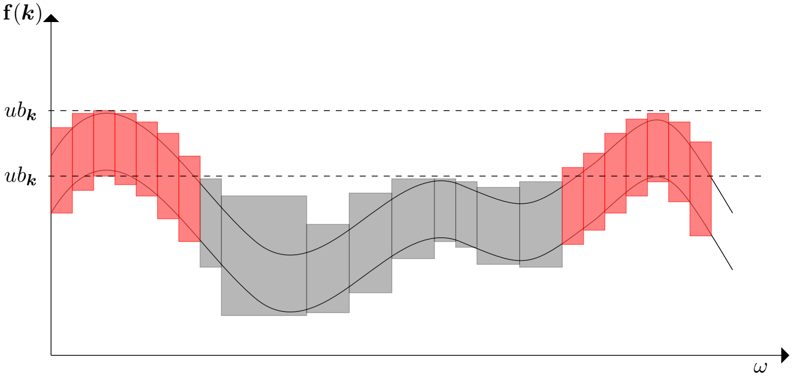


Secondary Branch and Bound algorithm: maximization



Secondary Branch and Bound algorithm: maximization



Inclusion properties

Let be $\boldsymbol{k} \subseteq \mathbb{K}$, we denote

$$\mathbf{f}_{max}(\boldsymbol{k}) = \{\sup_{\omega} f(k,\omega), k \in \boldsymbol{k}\}$$

$$\boldsymbol{\omega}_{\boldsymbol{k},max} = \{\omega \in [\omega] | \exists k \in \boldsymbol{k},\ \omega \text{ maximize } f(k,\omega)\}$$

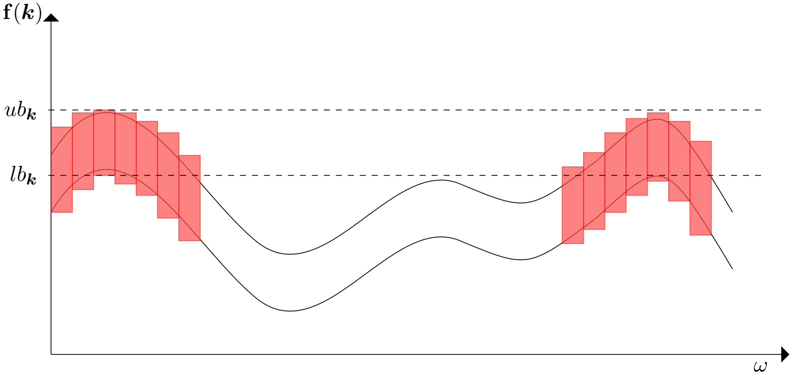
Let be $\boldsymbol{k}_1 \subseteq \boldsymbol{k}$.

Proposition

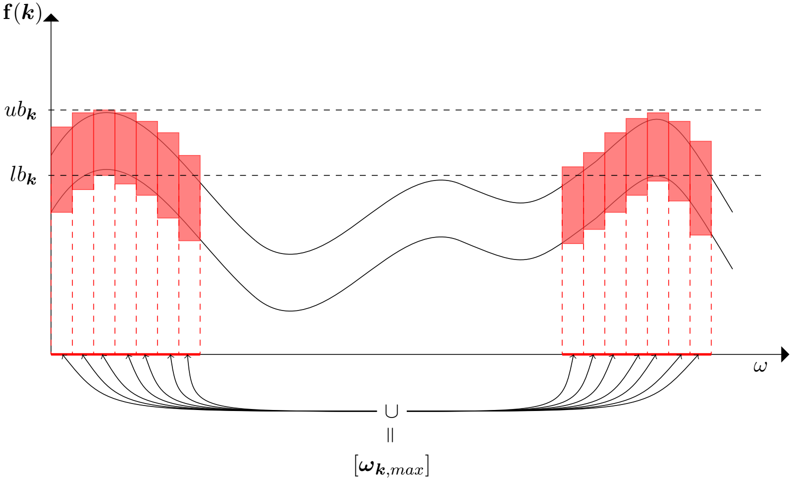
- $\mathbf{f}_{max}(\boldsymbol{k}_1) \subseteq \mathbf{f}_{max}(\boldsymbol{k})$
- $\boldsymbol{\omega}_{\boldsymbol{k}_1,max} \subseteq \boldsymbol{\omega}_{\boldsymbol{k},max}$

→These properties are used for each sub-box of B&B algorithm.

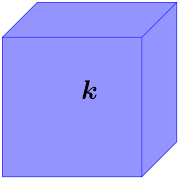
Inclusion properties



Inclusion properties



Main Branch and bound with inclusion properties



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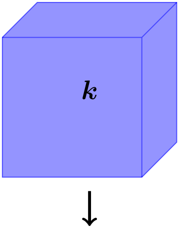
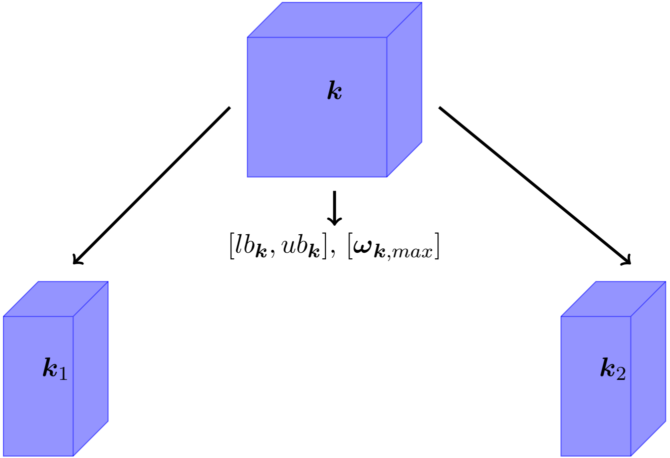


$$[lb_{\boldsymbol{k}}, ub_{\boldsymbol{k}}],\ [\boldsymbol{\omega}_{\boldsymbol{k},max}]$$

Main Branch and bound with inclusion properties



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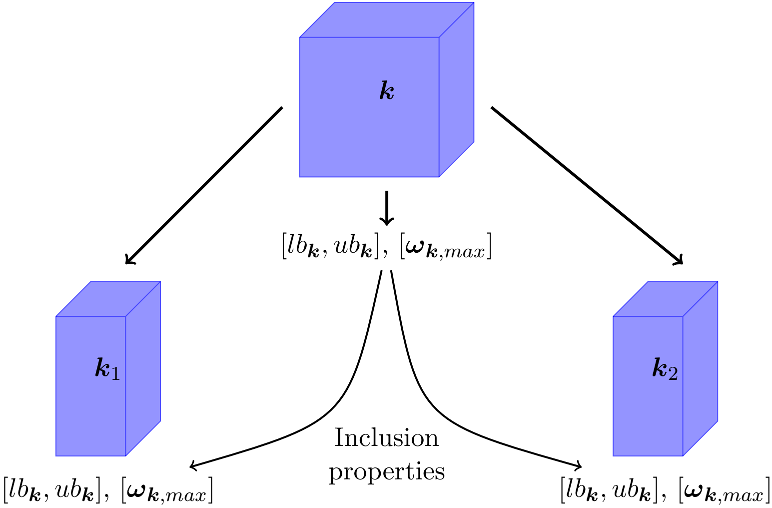

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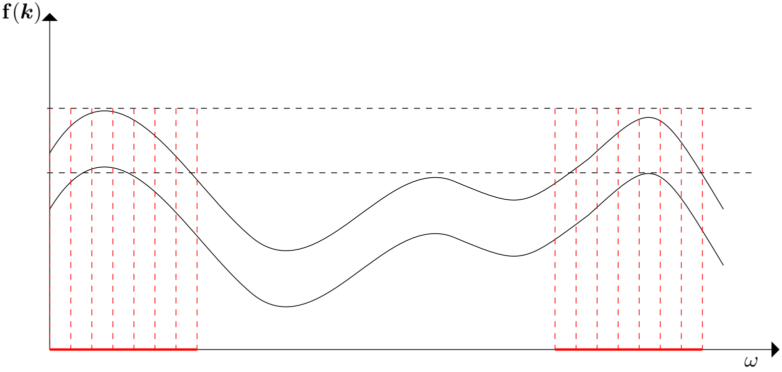

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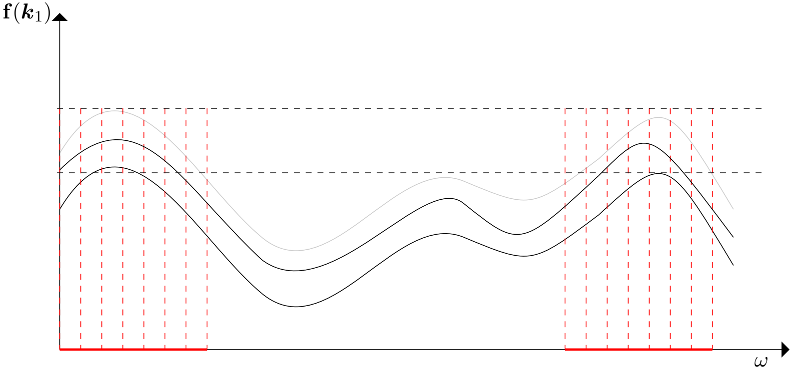

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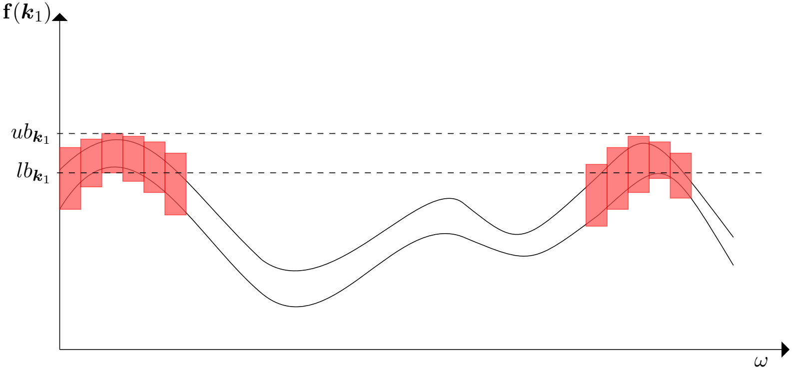

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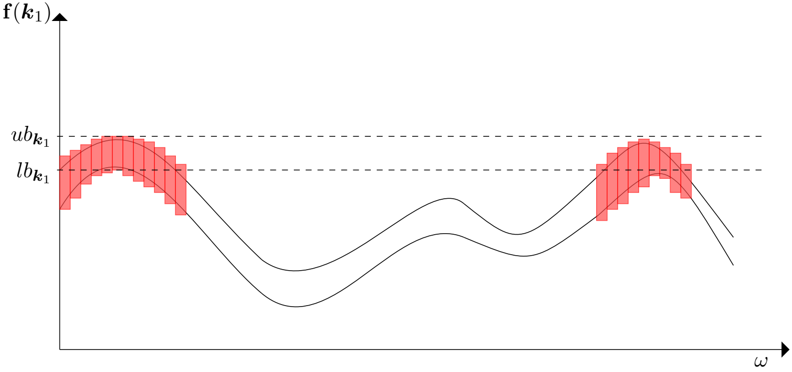

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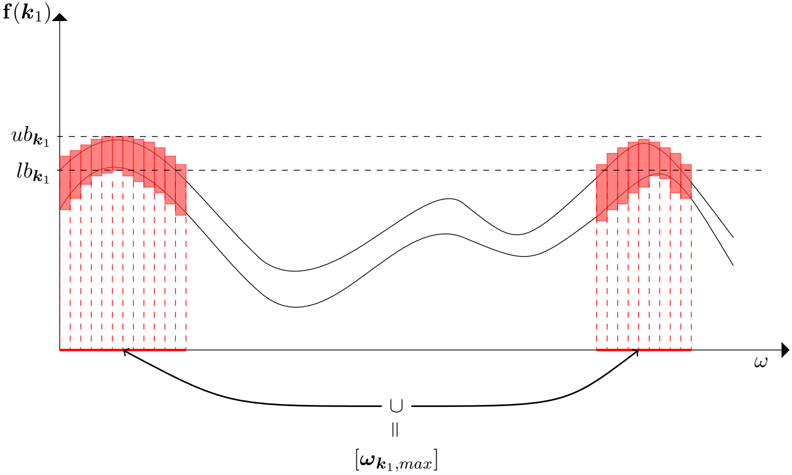

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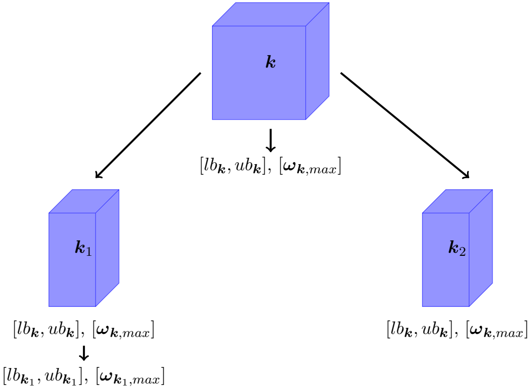

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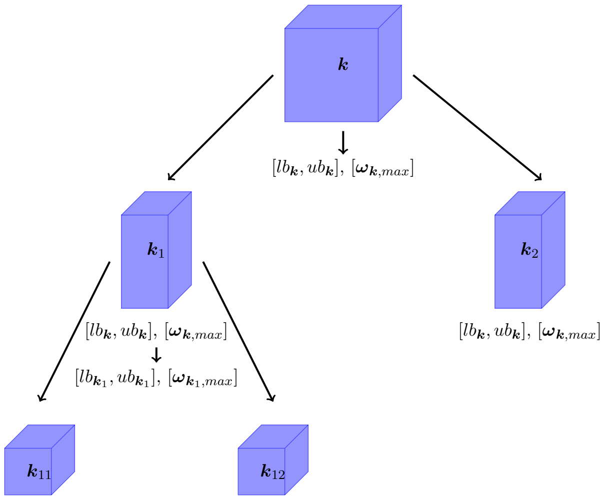

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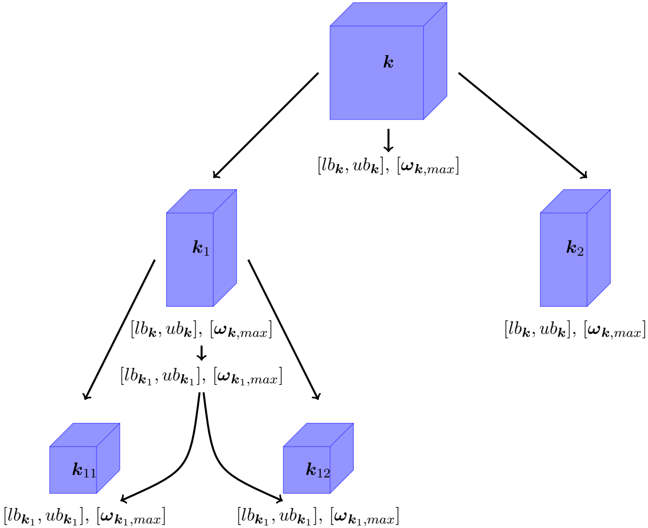


Plan

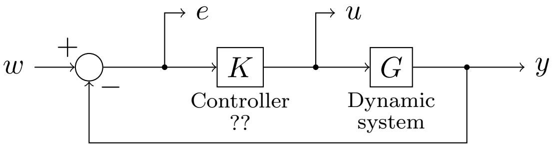
Problem under study

We consider a second-order system controlled by a PID, with contraints on error and control signals:

$$G(s) = \frac{1}{s^2 + 1.4s + 1}, \quad K(s) = k_p + \frac{k_i}{s} + \frac{k_d s}{1+s}.$$

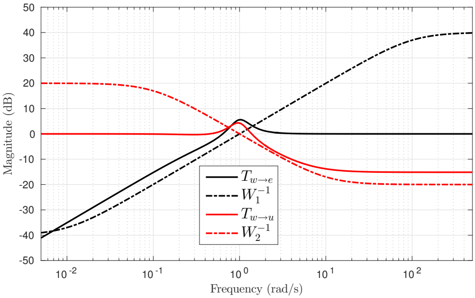
$$W_1(s) = \frac{s+100}{100s+1}, \qquad W_2(s) = \frac{10s+1}{s+10}.$$

Initial set of coefficient: $\mathbb{K} = \begin{pmatrix} [-10, 10] \\ [-10, 10] \\ [-10, 10] \end{pmatrix}$, $[\omega] = [10^{-2}, 10^{2}]$



Results

- In 13 mn, we proved that $\min_k \sup_\omega f(k,\omega) > 1$.
- Enclosure found in 1h 15 min:
  $\min_k \sup_\omega f(k,\omega) \in [1.5941, 1.9922]$, reached with
  $k_p = 0.584382$ $k_i = 0.566217$ and $k_d = -0.409415$.



Plan

Conclusion

- Taking advantage of Inclusion properties save computation time.
- We proposed a new approach to H_∞ synthesis problem.
- We can prove that the problem is not feasible in a **guaranteed way**.