A global optimization approach to H_∞ synthesis with parametric uncertainties applied to AUV control

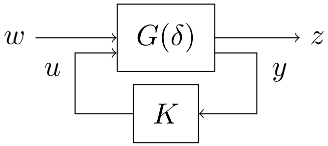
Dominique Monnet, Jordan Ninin, Benoît Clement
LAB-STICC, UMR 6285 / ENSTA-Bretagne

July 11, 2017

Introduction

LTI system: G

Regulation scheme:



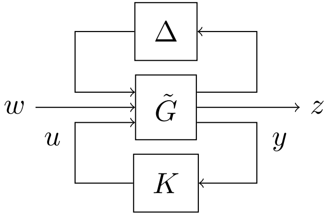
Uncertain parameters δ, $G \to G(\delta)$.

Robust Synthesis Problem

Find K structured, $\begin{cases} \|T_{w\to z}(\delta)\|_\infty \leq 1, & \forall \delta \in \boldsymbol{\delta} \\ \text{K stabilizes the closed loop,} & \forall \delta \in \boldsymbol{\delta} \end{cases}$

μ-analysis/ structured singular value

Problem modelization for μ synthesis/analysis:
$G(\delta) \to F_u(\tilde{G}, \Delta)$, $||\Delta||_\infty \leq 1$.

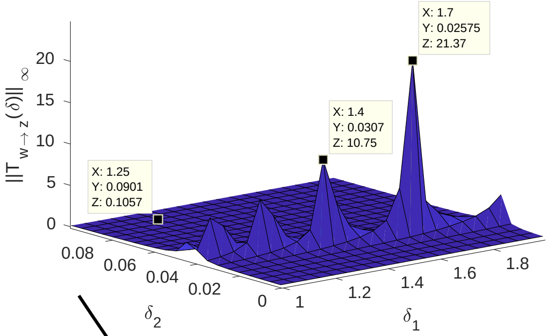


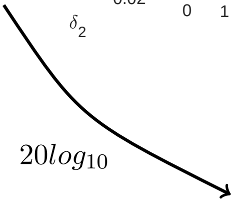
Theorem

$$\sup_{\omega>0} \mu_\Delta(Fl(\tilde{G}, K)) \leq 1 \iff \begin{cases} ||T_{w\to z}||_\infty \leq 1 \\ \text{Internal stability} \end{cases}$$

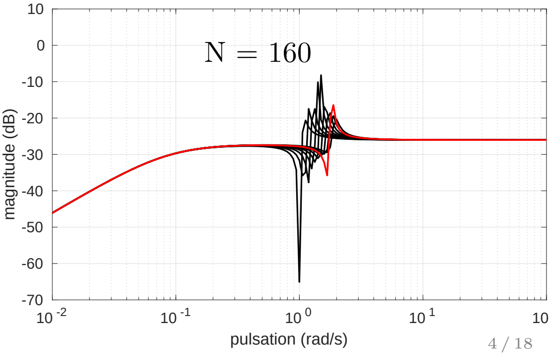
$$\sup_{\omega>0} \mu_\Delta(Fl(\tilde{G}, K)) \approx \max_{\omega_i, i=1\ldots N} \mu_\Delta(Fl(\tilde{G}(j\omega_i), K(j\omega_i))$$

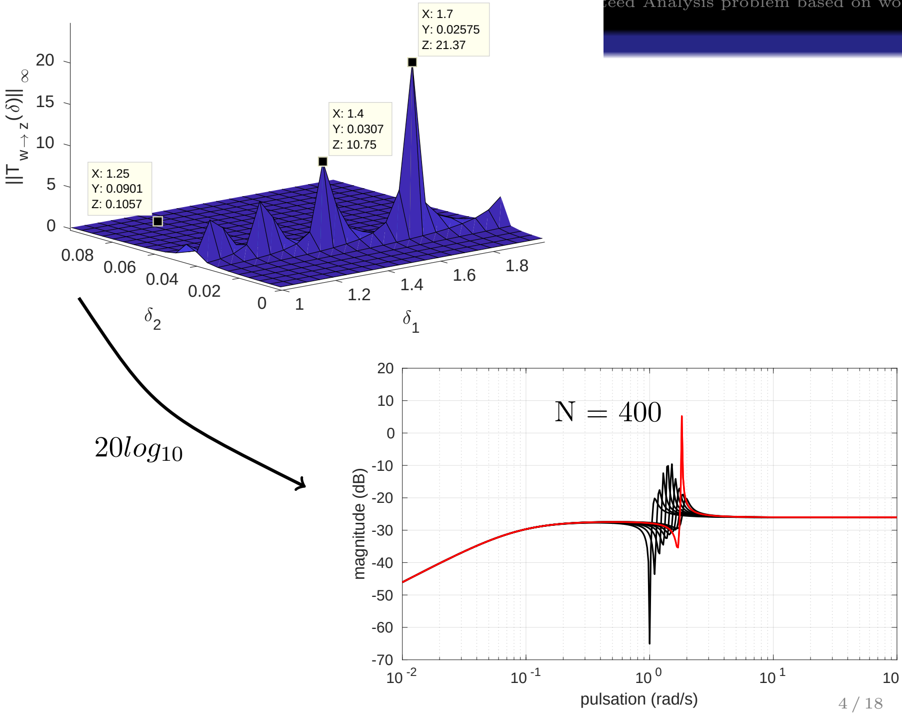
Discretization $\implies$ not guaranteed!

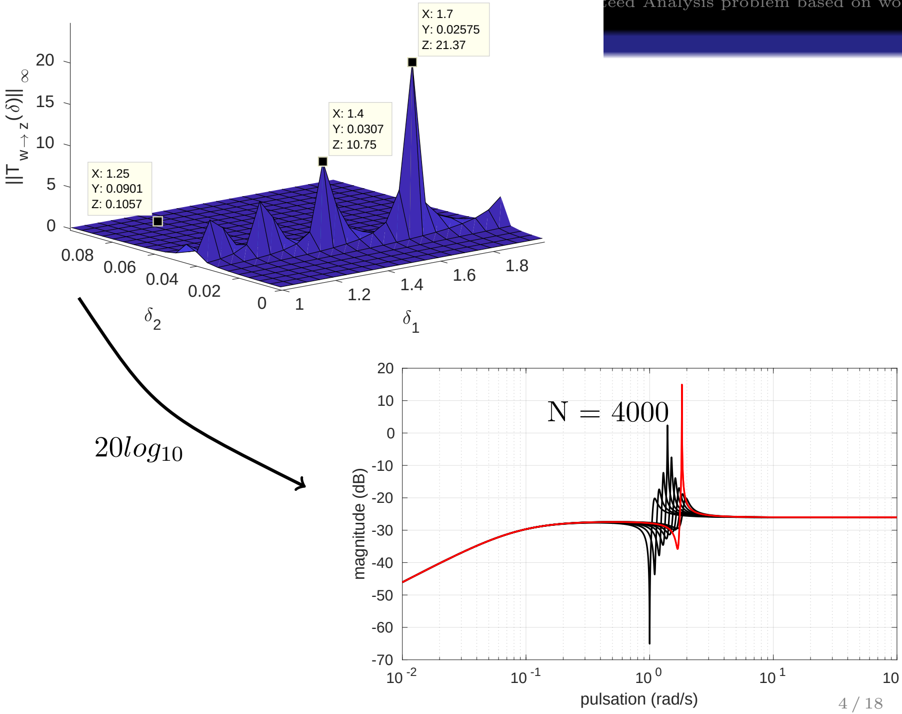

X: 1.7
Y: 0.02575
Z: 21.37
X: 1.4
Y: 0.0307
Z: 10.75
X: 1.25
Y: 0.0901
Z: 0.1057
$||T_{w\to z}(\delta)||_\infty$
20
15
10
5
0
0.08
0.06
0.04
0.02
0
δ_2
1
1.2
1.4
1.6
1.8
δ_1

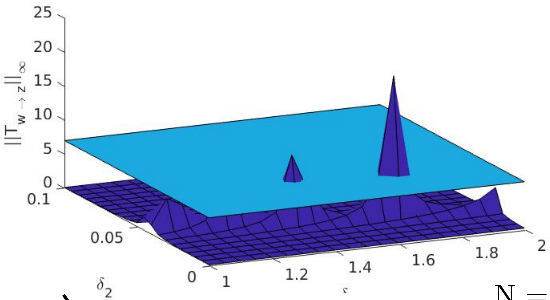


$20log_{10}$


N = 160
magnitude (dB)
10
0
-10
-20
-30
-40
-50
-60
-70
10^{-2}
10^{-1}
10^{0}
10^{1}
10
pulsation (rad/s)

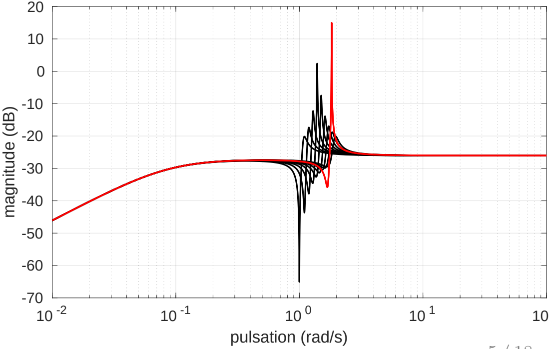

X: 1.7
Y: 0.02575
Z: 21.37
X: 1.4
Y: 0.0307
Z: 10.75
X: 1.25
Y: 0.0901
Z: 0.1057
$\|T_{w\to z}(\delta)\|_\infty$
δ_2
δ_1
$20log_{10}$
N = 400
magnitude (dB)
pulsation (rad/s)


X: 1.7
Y: 0.02575
Z: 21.37
X: 1.4
Y: 0.0307
Z: 10.75
X: 1.25
Y: 0.0901
Z: 0.1057
$\|T_{w\to z}(\delta)\|_\infty$
δ_2
δ_1
$20log_{10}$
N = 4000
magnitude (dB)
pulsation (rad/s)



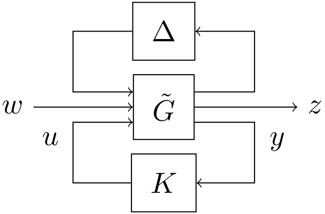
$20log_{10}$

N = 4000

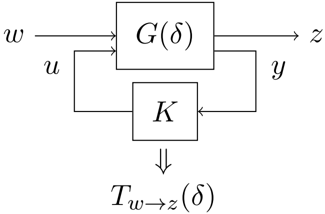


Dealing with uncertainties

μ analysis:



Global approach:



Worst case reformulation for CSP.

Problem

Does the controller K stabilizes the closed loop and ensure the H_∞ performances?

$$\begin{cases} ||T_{w\to z}(\delta)||_\infty \leq 1, & \forall\delta \in \boldsymbol{\delta} \\ T_{w\to z}(\delta) \text{ stable}, & \forall\delta \in \boldsymbol{\delta} \end{cases}$$

or

$$\begin{cases} ||T_{w\to z}(\delta)||_\infty \leq 1, & \forall\delta \in \boldsymbol{\delta} \\ R(\delta) \leq 0, & \forall\delta \in \boldsymbol{\delta} \text{ (Routh, polynomial inequalities)} \end{cases}$$

$$\Longrightarrow \text{ Interval Analysis}$$

Interval Analysis

Definition: Interval

An interval $\boldsymbol{x} = [\underline{x}, \overline{x}]$ is a close connected of $\mathbb{R}$, $\mathbb{IR}$ is the set of intervals.

Common operators $(+, -, \times, /, sin, cos, max, ...)$ can be extended to $\mathbb{IR}$

- $[1,2] + [-1,2] = [0,4]$
- $[1,2] - [-1,2] = [-1,3]$
- $[1,2] \times [-1,2] = [-2,4]$

Definition: Inclusion function

Let be $f : \mathbb{R}^n \mapsto \mathbb{R}^m$, $[f] : \mathbb{IR}^n \mapsto \mathbb{IR}^m$ is an inclusion function of f iff $\forall \boldsymbol{x} \in \mathbb{IR}^n,\ f(\boldsymbol{x}) = \{f(x),\ x \in \boldsymbol{x}\} \subseteq [f](\boldsymbol{x})$

Worst-case over frequency and uncertainty

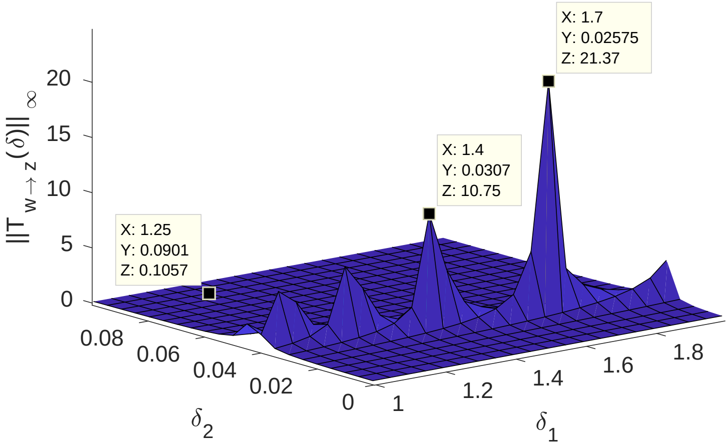
Example: SISO case $(dim(w) = dim(z) = 1)$.

$$||T_{w \to z}(\delta)||_\infty = \sup_{\omega \in \Omega} |T_{w \to z}(\delta, j\omega)|$$

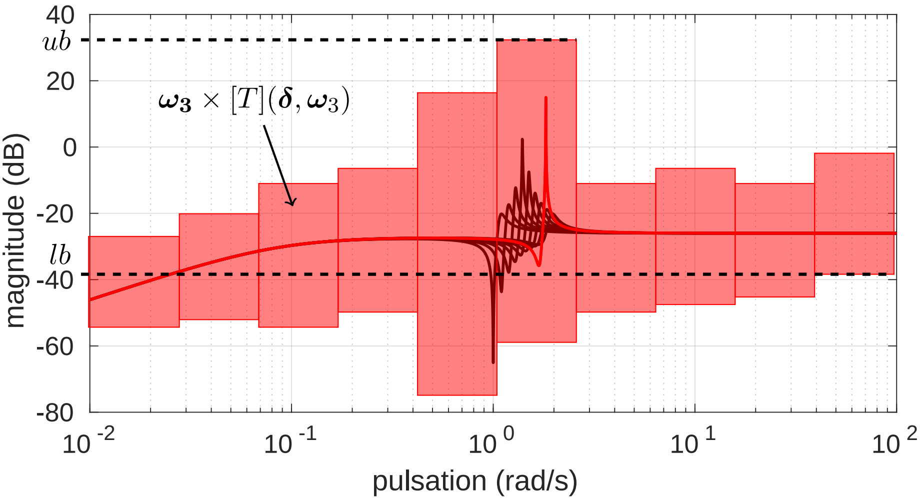
$$||T_{w \to z}(\delta)||_\infty \leq 1, \ \forall \delta \in \boldsymbol{\delta} \iff \sup_{\delta \in \boldsymbol{\delta}, \, \omega \in \Omega} |T_{w \to z}(\delta, j\omega)| \leq 1$$

$|T_{w \to z}(\delta, j\omega)|$ is a rational expression depending on δ and ω
$\to [T]$ inclusion function of $|T_{w \to z}(\delta, j\omega)|$

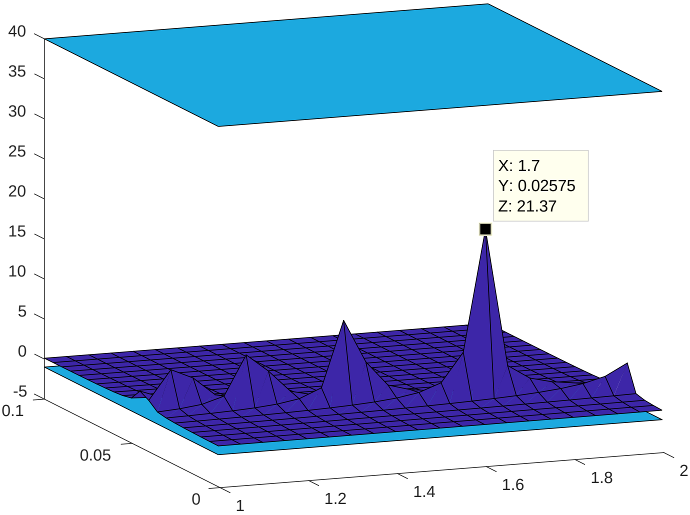
Example: 2 uncertain parameters



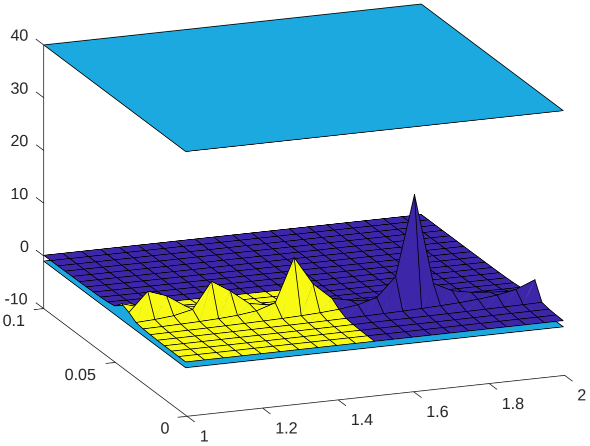
Example: 2 uncertain parameters



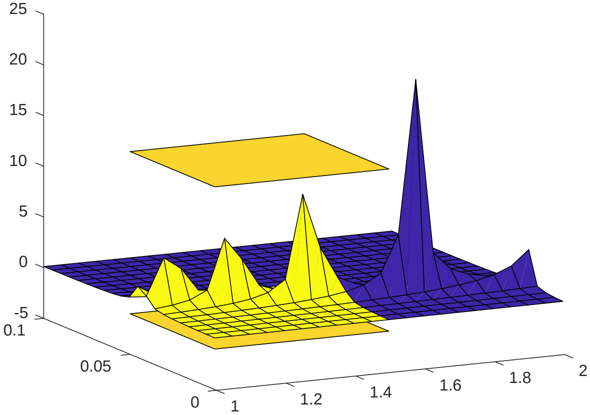
Example: 2 uncertain parameters



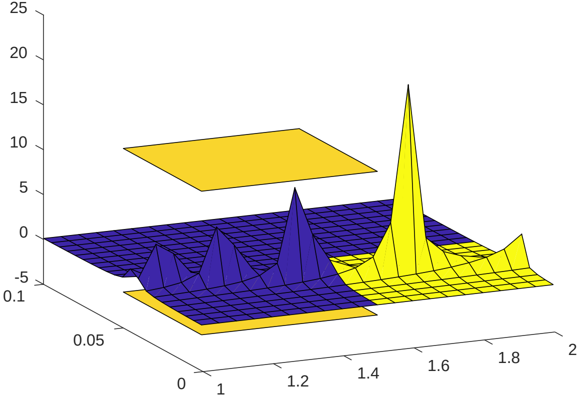
Example: 2 uncertain parameters



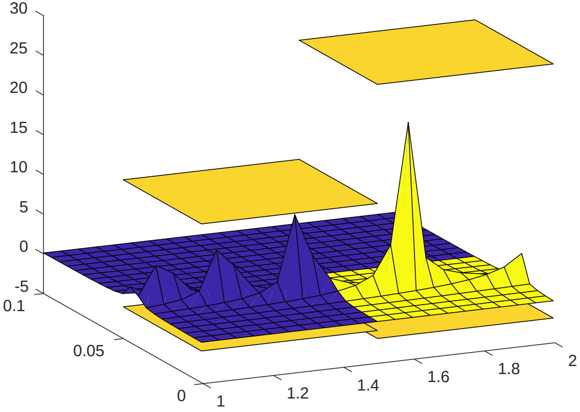
Example: 2 uncertain parameters



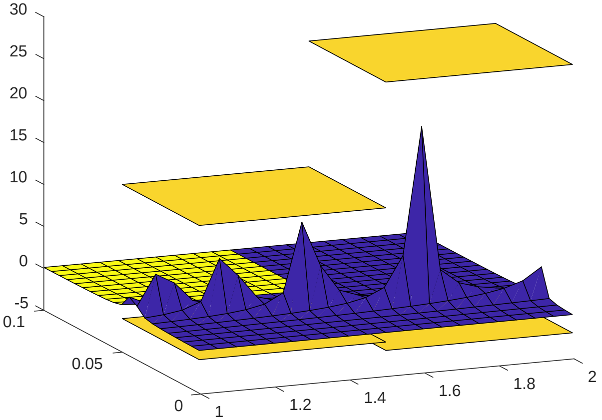
Example: 2 uncertain parameters



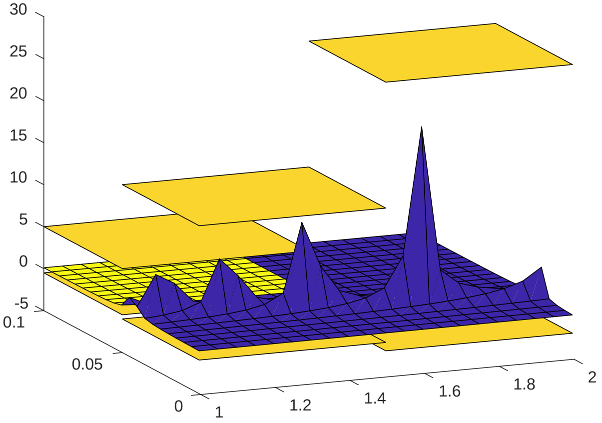
Example: 2 uncertain parameters



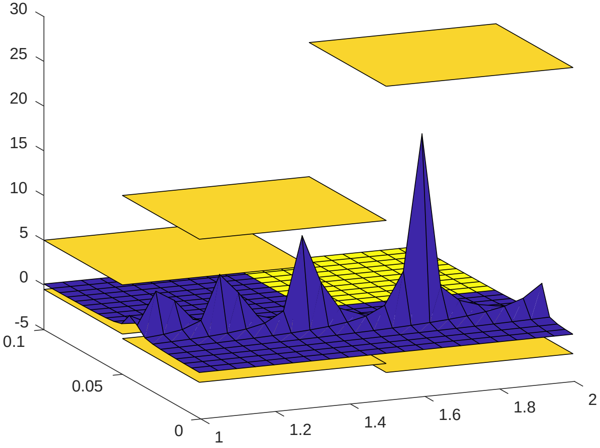
Example: 2 uncertain parameters



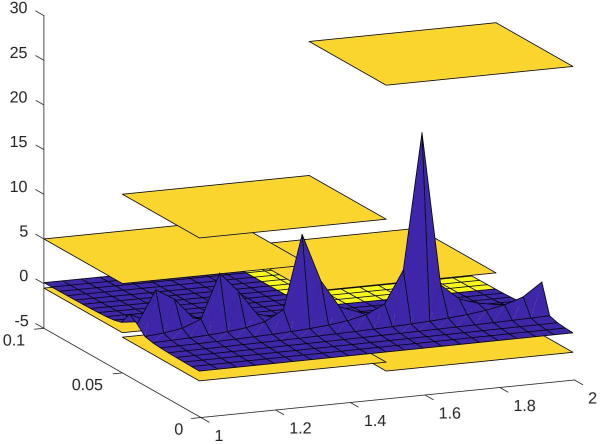
Example: 2 uncertain parameters



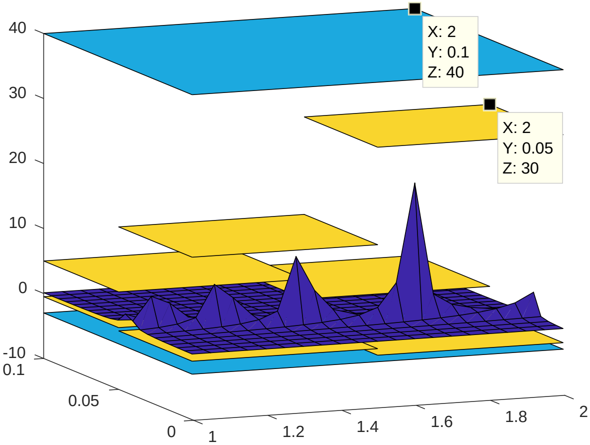
Example: 2 uncertain parameters



Example: 2 uncertain parameters



Example: 2 uncertain parameters



Robust Performance:

$$\sup_{\delta \in \boldsymbol{\delta}} ||T_{w \to z}(\delta)||_\infty \subseteq [lb, ub]$$

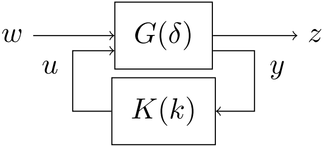
$\rightarrow$ converges to the global optimum.

- $ub \leq 1 \implies$ robust performance, i.e., performance $\forall \delta \in \boldsymbol{\delta}$.
- $lb > 1 \implies$ not robust.
- $1 \in [ub, lb] \implies$?

Robust Stability:

- $\overline{[R](\boldsymbol{\delta})} \leq 0 \implies$ robust stability, i.e., stability $\forall \delta \in \boldsymbol{\delta}$.
- $\underline{[R](\boldsymbol{\delta})} > 0 \implies$ instability.
- $0 \in [R](\boldsymbol{\delta}) \implies$?

Structured controller: $K \to K(k)$,
$K(k) = k_p + \frac{k_i}{s} + k_d s, \quad k = (k_p, k_i, k_d)^T$



Problem: worst case minimization

$$\begin{cases} \min_{k} \sup_{\delta \in \boldsymbol{\delta}} ||T_{w \to z}(k, \delta)||_\infty \\ \text{subject to } T_{w \to z}(k, \delta) \text{ stable.} \end{cases}$$

Global optimization: compute a guaranteed enclosure $[Lb, Ub]$ of the global minimum of over a box $\boldsymbol{k}$.

Analysis over a set of controller

Consider a set of controller $K(\boldsymbol{k})$.

Robust Stability over $\boldsymbol{k}$:

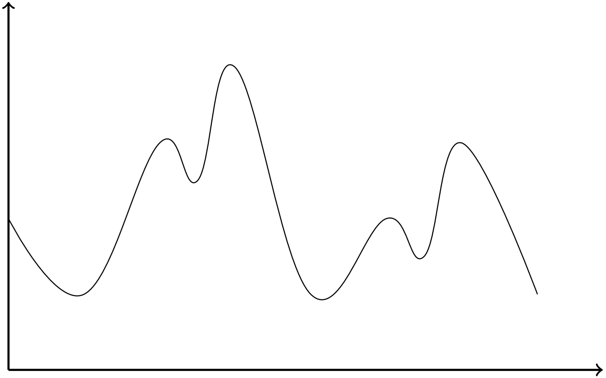
- $\overline{[R](\boldsymbol{k},\boldsymbol{\delta})} \leq 0 \implies$ robust stability $\forall k \in \boldsymbol{k}$.
- $\underline{[R](\boldsymbol{k},\boldsymbol{\delta})} > 0 \implies$ instable $\forall k \in \boldsymbol{k}$.
- $0 \in [R](\boldsymbol{k},\boldsymbol{\delta}) = 0 \implies$?.

Enclosure of objective function over $\boldsymbol{k}$:

- $\left\{ \sup_{\delta \in \boldsymbol{\delta}} ||T_{w \to z}(k,\delta)||_\infty,\ k \in \boldsymbol{k} \right\} \subseteq [lb_{\boldsymbol{k}}, ub_{\boldsymbol{k}}]$

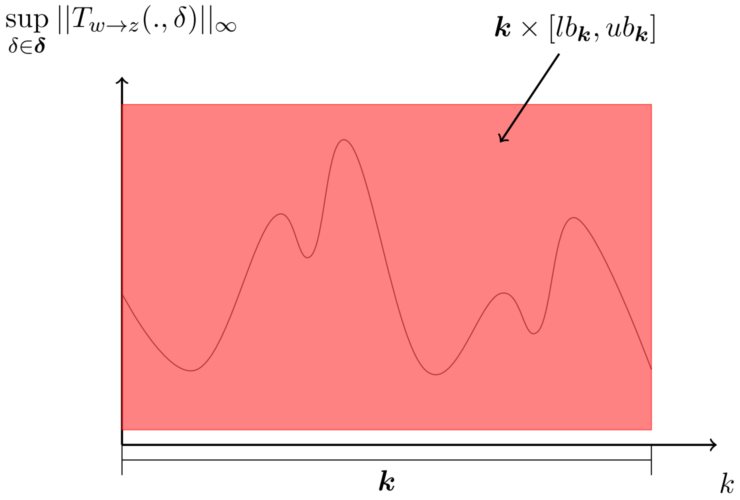
Branch and Bound Algorithm

$\sup_{\delta \in \boldsymbol{\delta}} ||T_{w \to z}(., \delta)||_\infty$



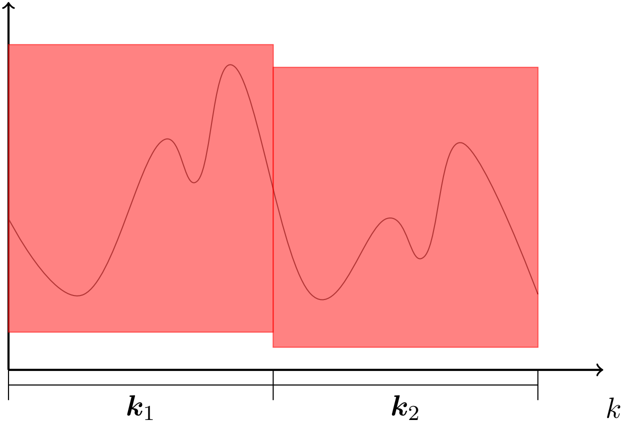
k

Branch and Bound Algorithm



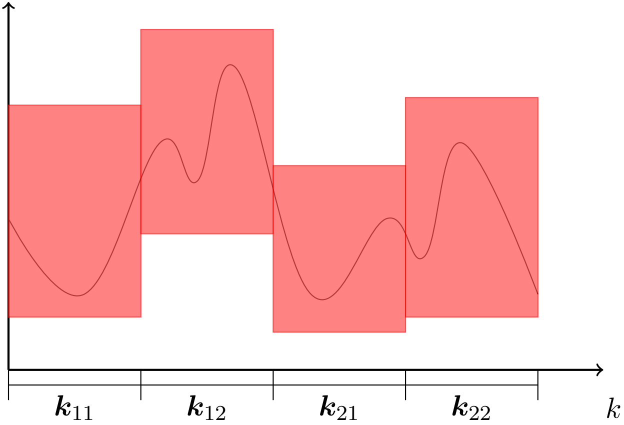
Branch and Bound Algorithm

$$\sup_{\delta \in \boldsymbol{\delta}} ||T_{w\to z}(., \delta)||_\infty$$

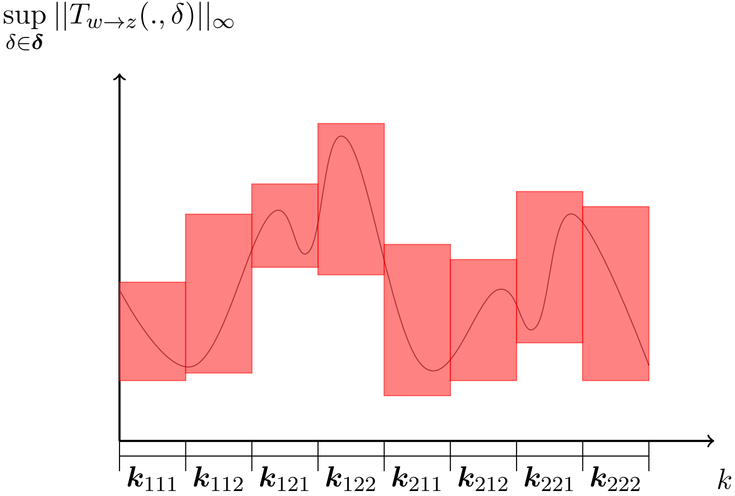


Branch and Bound Algorithm

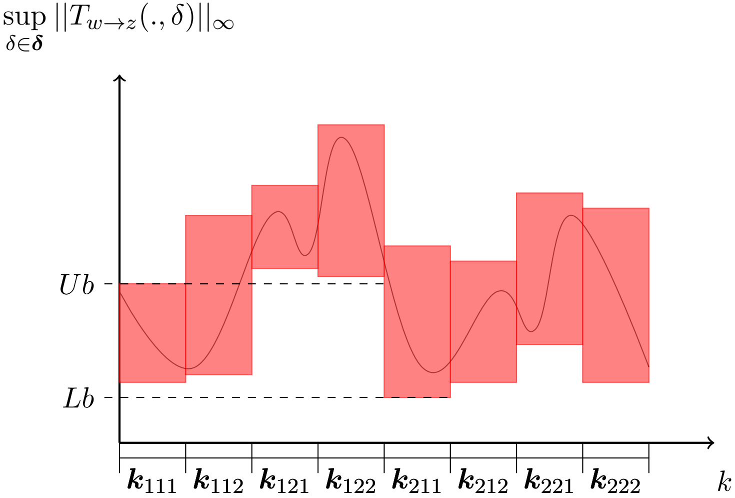
$$\sup_{\delta \in \boldsymbol{\delta}} ||T_{w \to z}(., \delta)||_\infty$$



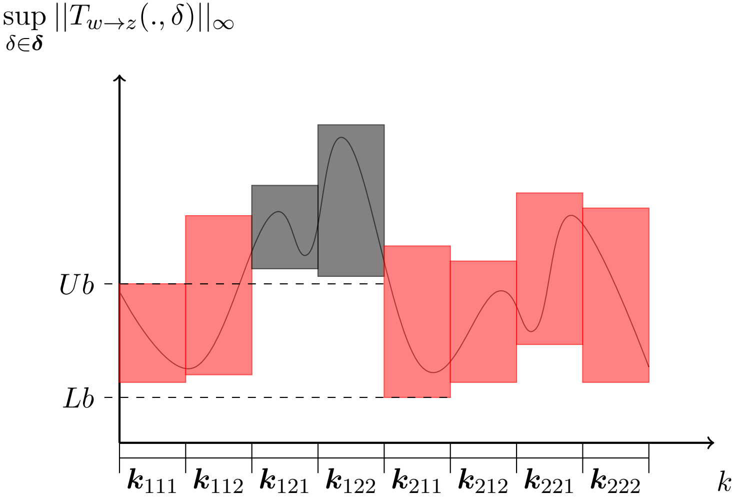
Branch and Bound Algorithm



Branch and Bound Algorithm

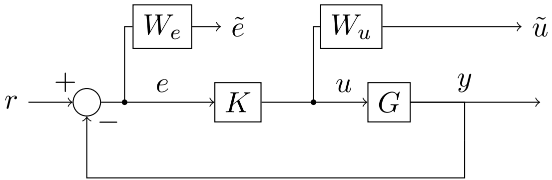


Branch and Bound Algorithm



AUV control

- AUV model: $G = \frac{1}{\delta_1 s^2 + \delta_2 s}$, $\delta_1 \in [0.3, 0.69]$, $\delta_2 \in [1.26, 2.34]$.
- Controller: $K = k_p + k_i + \frac{k_d s}{1+s}$, searched in $\boldsymbol{k} = [-5, 5] \times [-5, 5] \times [-5, 5]$.



Objectives:

- $||T_{r\to\tilde{e}}(k,\delta)||_\infty \leq 1$, with $W_e(s) = 0.5\frac{s+0.92}{s+0.0046}$.
- $||T_{r\to\tilde{u}}(k,\delta)||_\infty \leq 1$, with $W_u(s) = 0.01$.
- Internal stability.

Synthesis result

Problem: min-max formulation

$$\begin{cases} \min_{k \in \boldsymbol{k}} \left[\sup_{\delta} \left(\max\left(\|T_{r \to \tilde{e}}(k,\delta)\|_\infty, \|T_{r \to \tilde{u}}(k,\delta)\|_\infty \right) \right) \right] \\ \text{s.t. } K \text{ robustly stabilizes the closed loop.} \end{cases}$$

Results:

- Enclosure of the global minimum: $[0.66, 0.74]$
- Best controller found: $K(\tilde{k}) = 1.471 + \frac{0.103}{s} + \frac{1.471s}{1+s}$,
  $\sup_{\delta} \left(\max\left(\|T_{r \to \tilde{e}}(k,\delta)\|_\infty, \|T_{r \to \tilde{u}}(k,\delta)\|_\infty \right) \right) \leq 0.74.$
- CPU time: 10 mn.

Conclusion

- Guaranteed alternative to μ-Analysis.
- Works for SISO and MIMO systems.

- Converges to **the global minimum**, i.e. the *best robust* controller.

Interval Analysis $\Longrightarrow$ **No** conservatism.

Annex: MIMO case

How to compute $||T_{w\to z}||_\infty$, with $dim(w) > 1$ and $dim(z) > 1$?
Consider performance outputs separately:

$$\begin{aligned} T_{w\to z_j} &= (T_{w_1\to z_j}, ..., T_{w_n\to z_j}) \\ ||T_{w\to z_j}||_\infty &= \sup_\omega \sqrt{\lambda_{max}(T_{w\to z_j}(j\omega)T_{w\to z_j}(j\omega)^*)} \\ &= \sup_\omega \sqrt{\sum_{i=1}^n |T_{w_i\to z_j}(j\omega)|^2} \end{aligned}$$

Use $\max_j(||T_{w\to z_j}||_\infty)$ instead of $||T_{w\to z}||_\infty$ as objective.

$$\max_j(||T_{w\to z_j}||_\infty) \leq ||T_{w\to z}||_\infty$$