Global Optimization of continuous MinMax problem

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Min Max problems

Min Max problems appear in:

- Robust control
- Game theory
- Risk management
- Every problem involving uncertainty

Plan

What is control?

- Dynamic system (Robot, Missile, Dam, Washing machine...).

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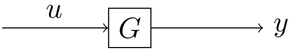
- Sensors (INS, Sonar, Temperature/Pressure sensor, ...).



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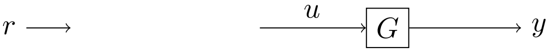
- Reference to follow.



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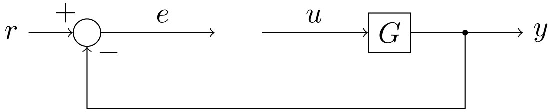

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- Actuators (Motor, Steering wheel, Flap, ...).
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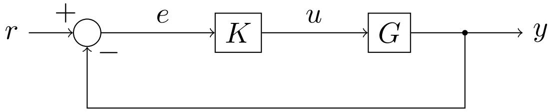
- Reference to follow.
- Controller to close the loop.



Frequency constraint

Frequency constraint on $e(i\omega)$: we want $|\frac{e(i\omega)}{r(i\omega)}| = |T_{r\to e}(K, i\omega)|$ to be small

$$\forall \omega \geq 0,\ |T_{r\to e}(K, i\omega)| \leq |W(i\omega)| \iff \sup_{\omega}(|T_{r\to e}(K, i\omega)W^{-1}(i\omega)|) \leq 1$$

Min max problem formulation

Stability constraint:

- The closed loop system is stable $\iff R(K) \leq 0$ (Routh criterion).
- $R(K) \leq 0$ is a non-convex rational system.

Problem formulation

$$\begin{cases} \min_{K} \sup_{\omega} |T_{r\to e}(K, i\omega)W^{-1}(i\omega)|, \\ s.t. \quad R(K) \leq 0 \end{cases}$$

We want:

- an enclosure of the minimum.
- reliable computation.

$\rightarrow$ Interval Based Branch and Bound Algorithm

Plan

Min max problem formulation

We search $x^* \in \mathcal{X}$ such that $\sup_{\boldsymbol{y} \in \mathcal{Y}} f(x^*, y)$ is minimal.

Constrained Min max problem

$$\begin{cases} \min_{x \in \mathcal{X}} \sup_{y \in \mathcal{Y}} f(x, y), \\ \\ s.t. \quad C_x(x) \leq 0 \\ \qquad\;\; C_{xy}(x, y) \leq 0 \end{cases}$$

- $\mathcal{X}$ and $\mathcal{Y}$ are bounded.
- f, $\mathcal{C}_x$ and $\mathcal{C}_{xy}$ can be evaluated with interval computation.

Main Branch and bound algorithm: minimization

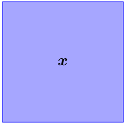
Interval Based Branch and Bound Algorithm

Init: push $\mathcal{X}$ in $\mathcal{L}$

1. Choose a box $\boldsymbol{x}$ from $\mathcal{L}$.
2. Contract $\boldsymbol{x}$ w.r.t $\mathcal{C}_x(\boldsymbol{x}) \leq 0$ using CSP techniques.
3. Compute $[lb_{\boldsymbol{x}}, ub_{\boldsymbol{x}}]$ an enclosure of $\sup_{y \in \mathcal{Y}} \mathbf{f}(\boldsymbol{x}, y)$.
4. Try to find a good feasible solution in $\boldsymbol{x}$.
5. Update best current solution.
6. Bisect $\boldsymbol{x}$ into $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, push $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$ in $\mathcal{L}$.

Stop criterion: $width([\min_{\boldsymbol{x} \in \mathcal{L}} lb_{\boldsymbol{x}}, \min_{\boldsymbol{x} \in \mathcal{L}, \mathcal{C}(\boldsymbol{x}) \leq 0} ub_{\boldsymbol{x}}]) \leq \epsilon$.

Main Branch and bound



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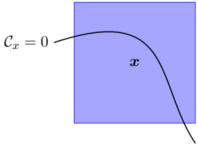

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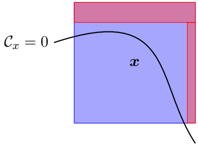

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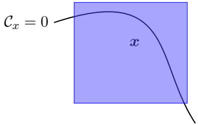
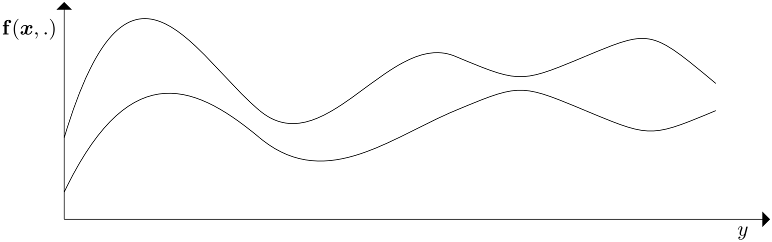


Secondary Branch and Bound algorithm: maximization



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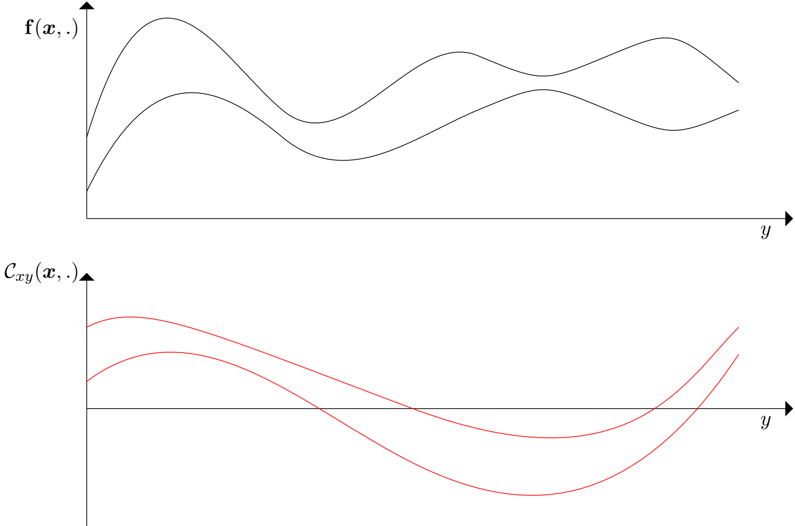

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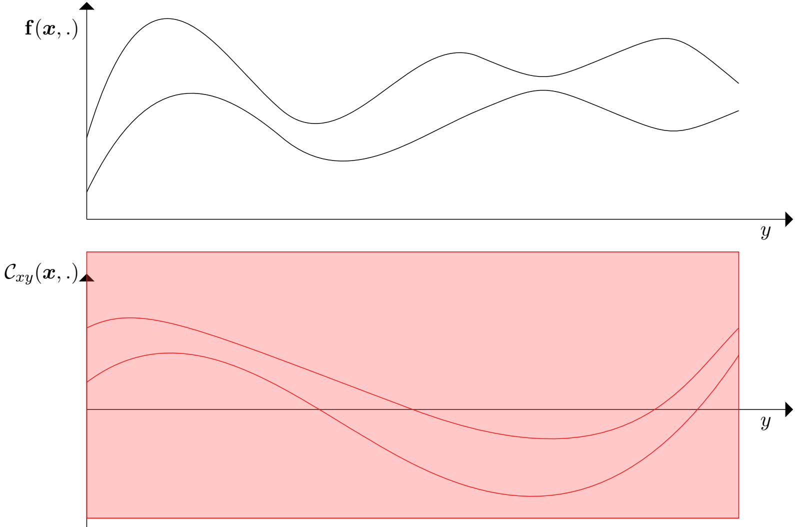

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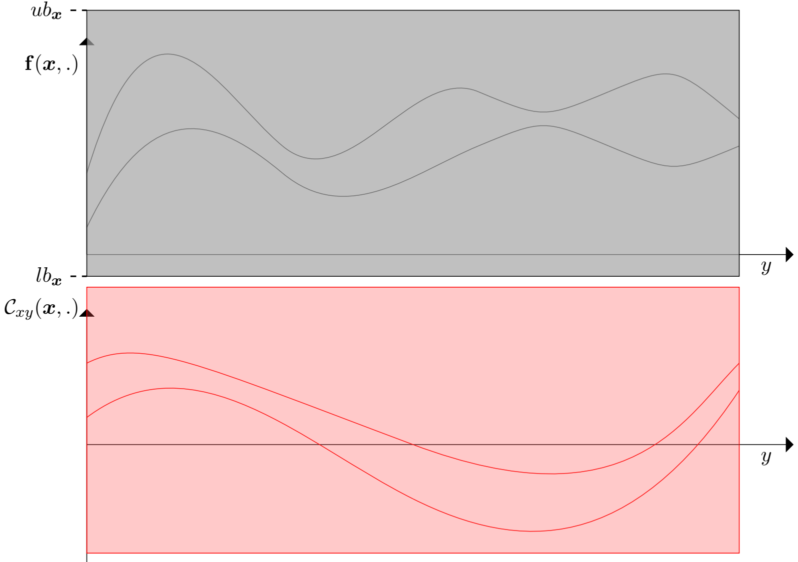

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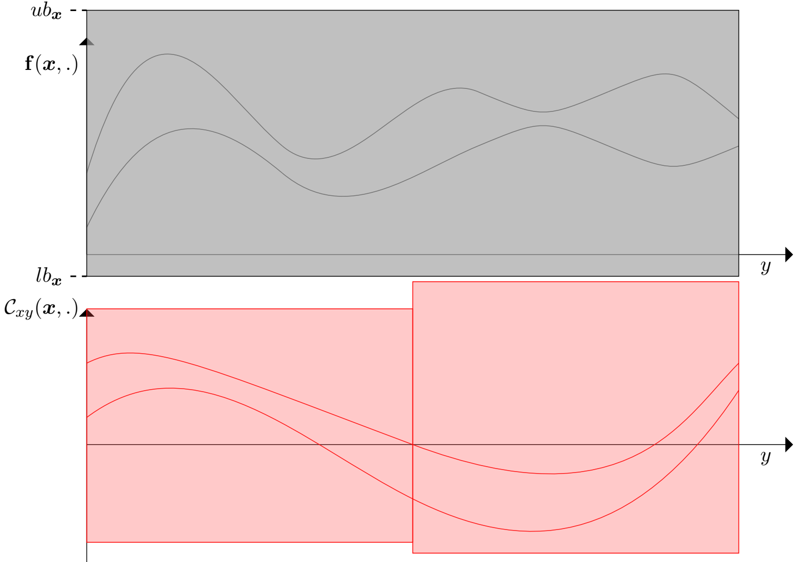

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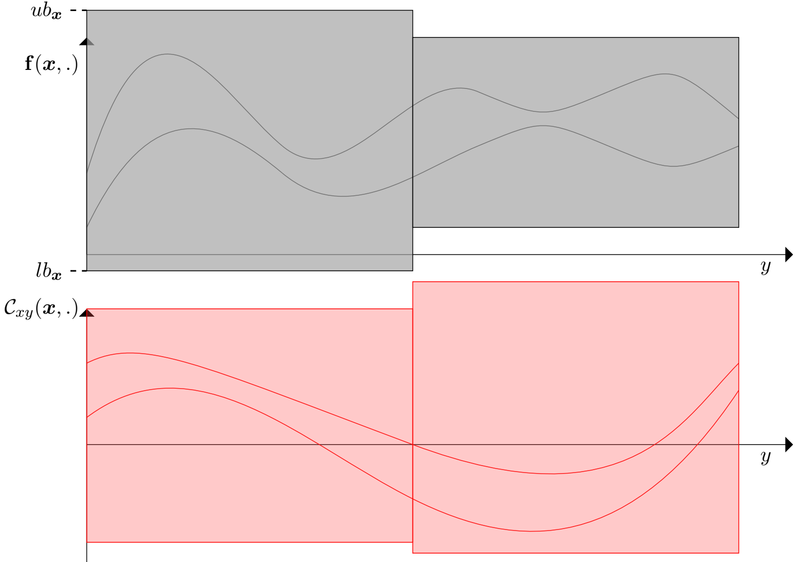

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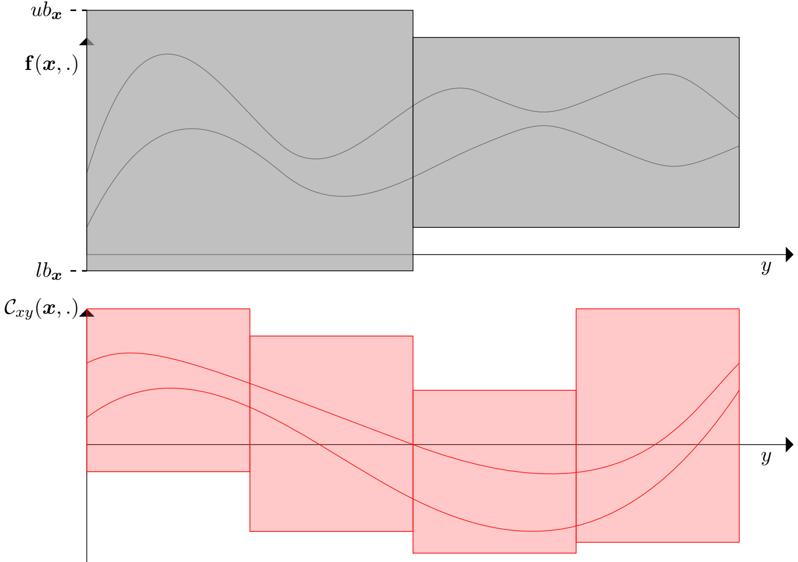

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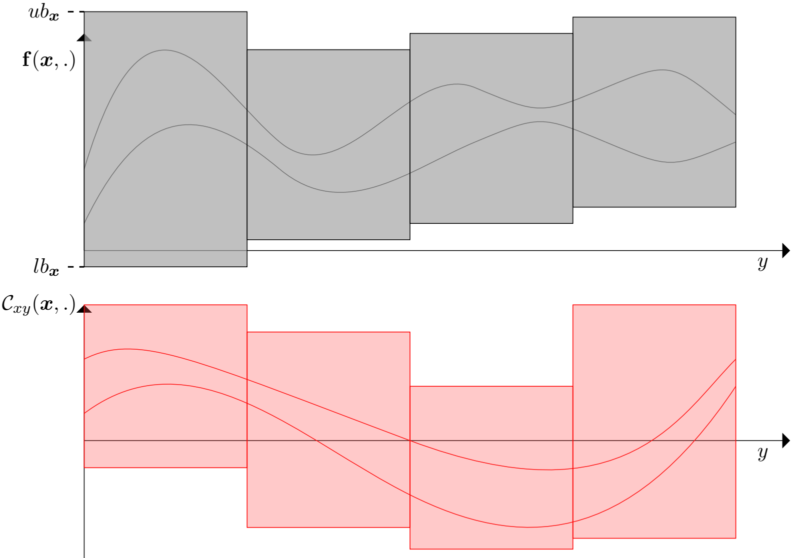

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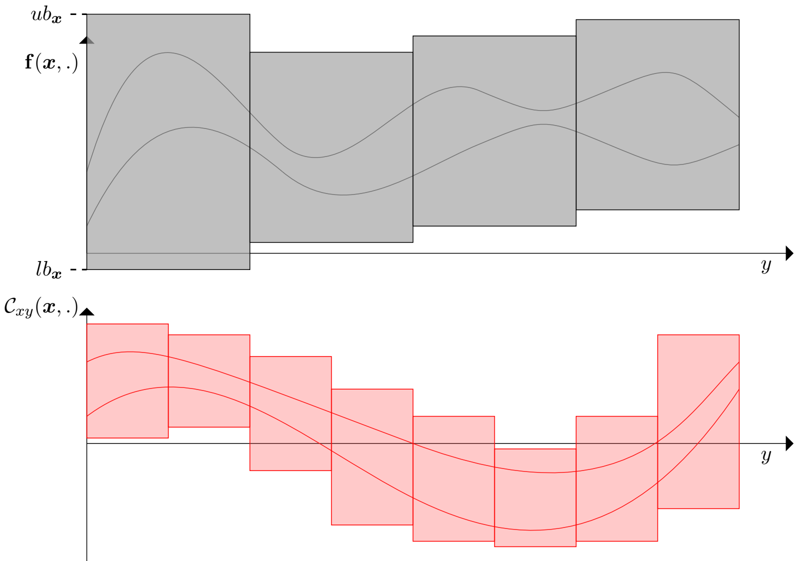

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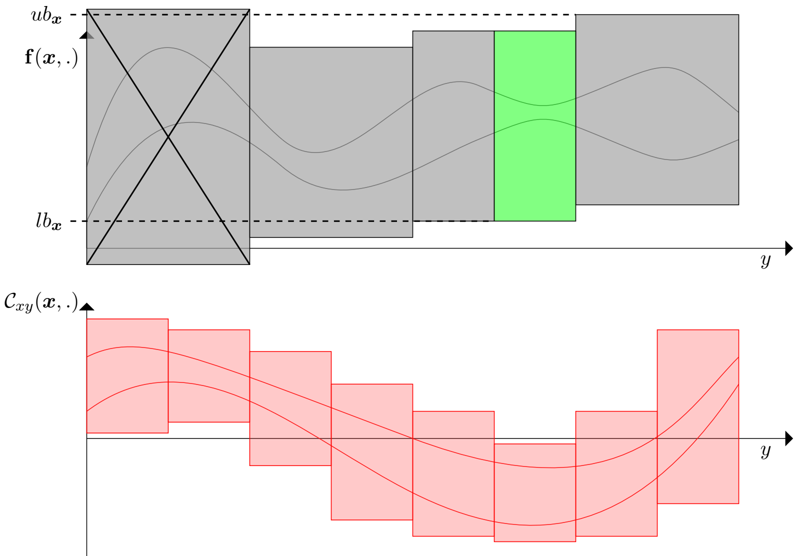

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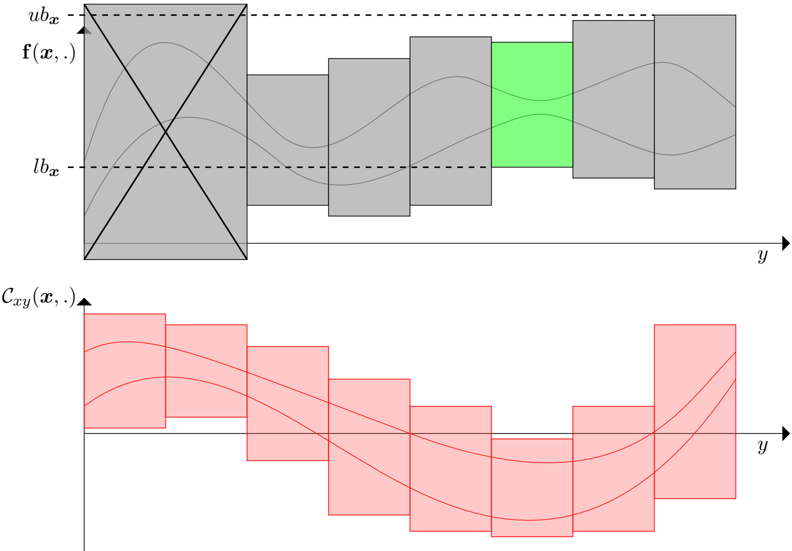

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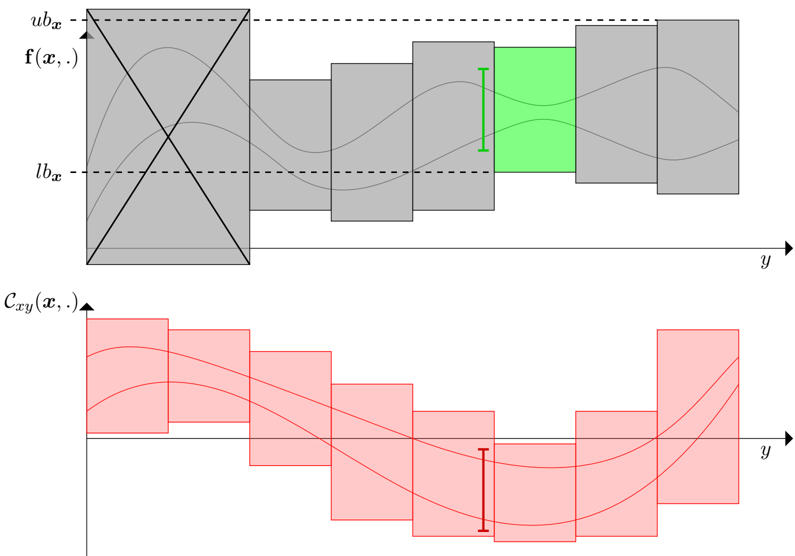

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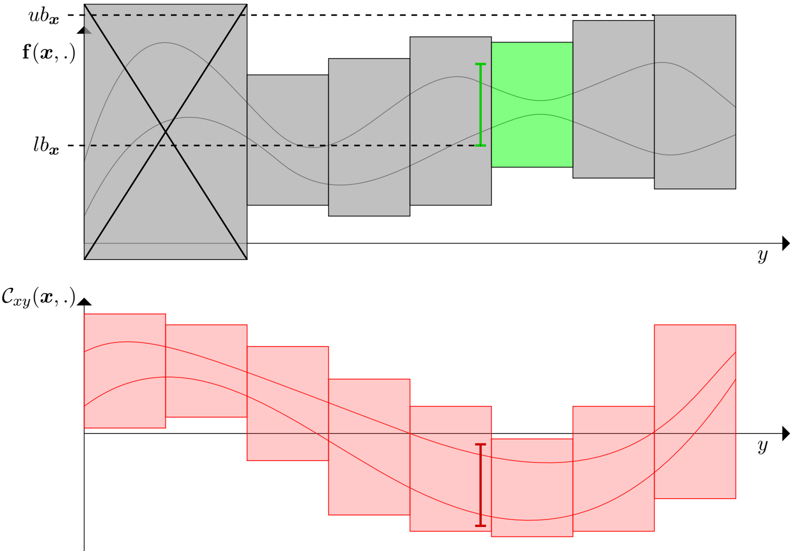

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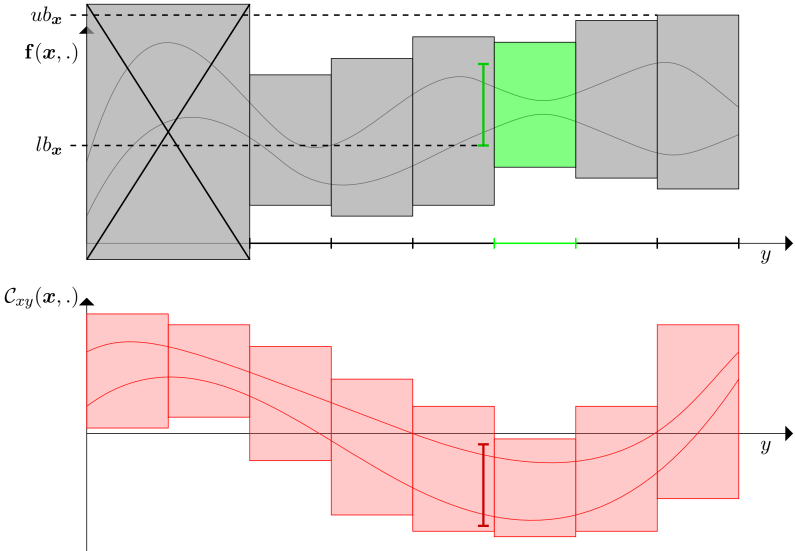

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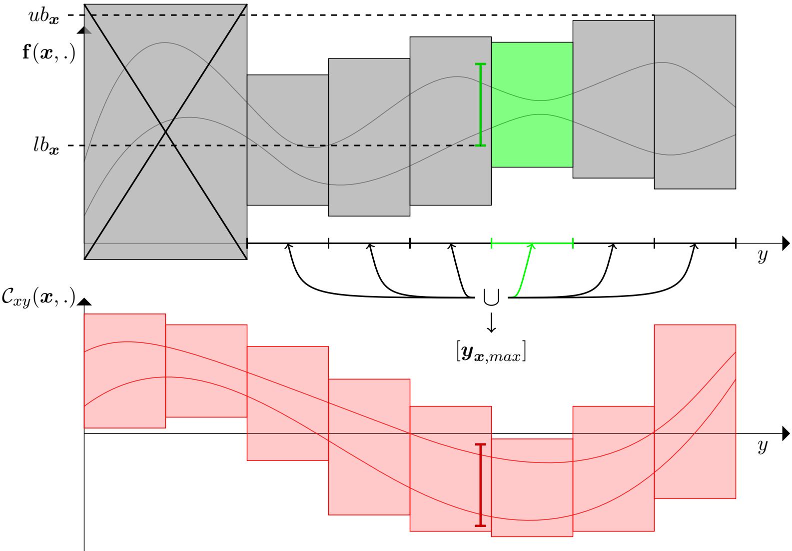


Inclusion properties

Let be $\boldsymbol{x} \subseteq \mathcal{X}$ and $\boldsymbol{y} \subseteq \mathcal{Y}$, we denote

$$\mathbf{f}_{max}(\boldsymbol{x}) = \{\sup_{y} f(x,y), x \in \boldsymbol{x}\}$$

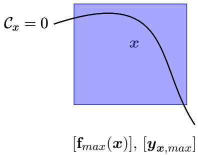
$$\boldsymbol{y}_{\boldsymbol{x},max} = \{y \in \mathcal{Y} | \exists x \in \boldsymbol{x},\ y \text{ maximizes } f(x,y)\}$$

Let be $\boldsymbol{x}_1 \subseteq \boldsymbol{x}$.

Proposition

- $\mathbf{f}_{max}(\boldsymbol{x}_1) \subseteq \mathbf{f}_{max}(\boldsymbol{x})$
- $\boldsymbol{y}_{\boldsymbol{x}_1,max} \subseteq \boldsymbol{y}_{\boldsymbol{x},max}$
- $\mathcal{C}_x(\boldsymbol{x}) \leq 0 \implies \mathcal{C}_x(\boldsymbol{x}_1) \leq 0$
- $\mathcal{C}_{xy}(\boldsymbol{x},\boldsymbol{y}) \leq 0 \implies \mathcal{C}_{xy}(\boldsymbol{x}_1,\boldsymbol{y}) \leq 0$

Main Branch and bound



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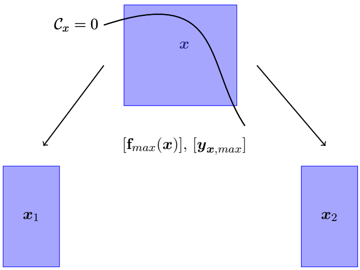

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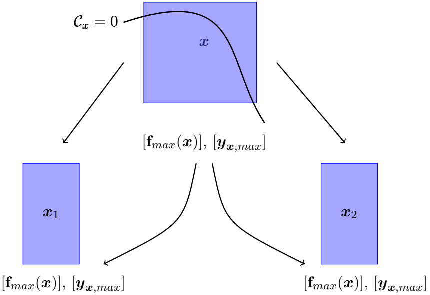

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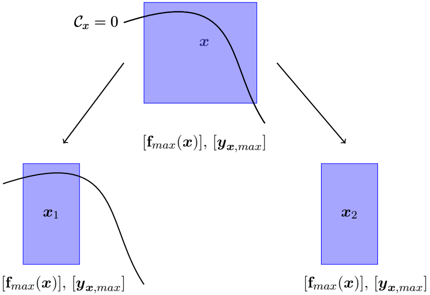

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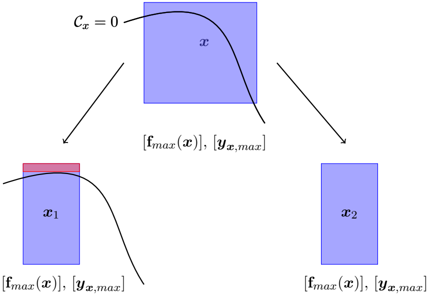

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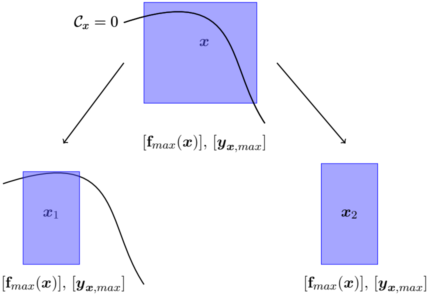
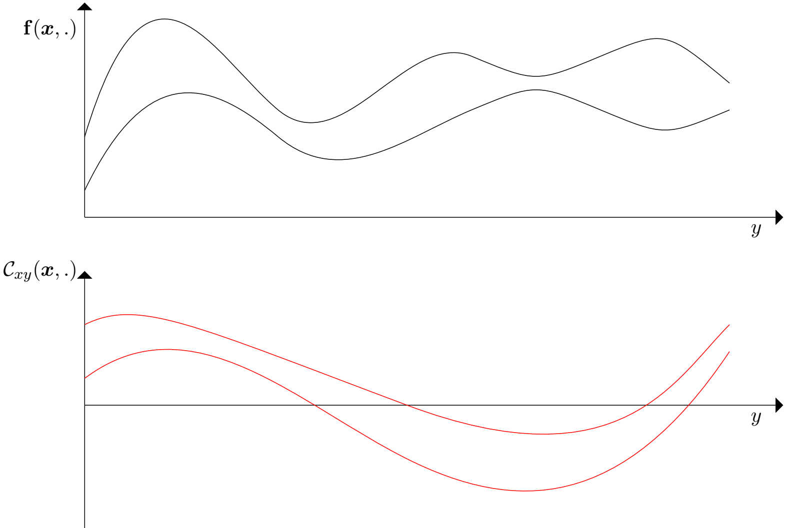


Main Branch and bound with Inclusion properties



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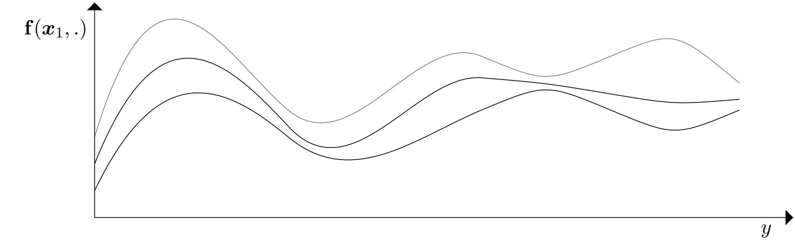

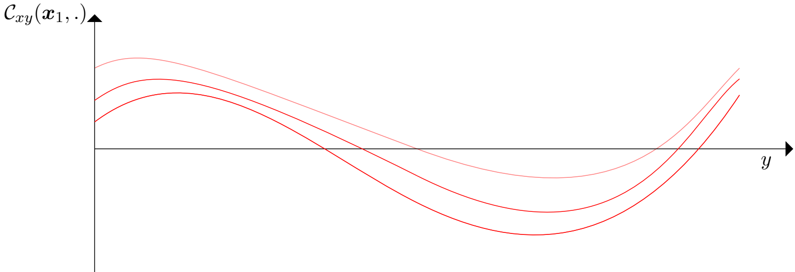

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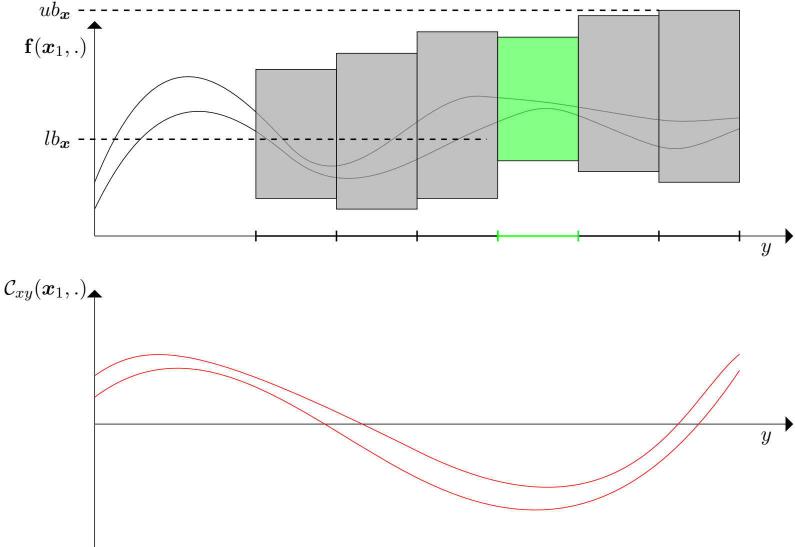

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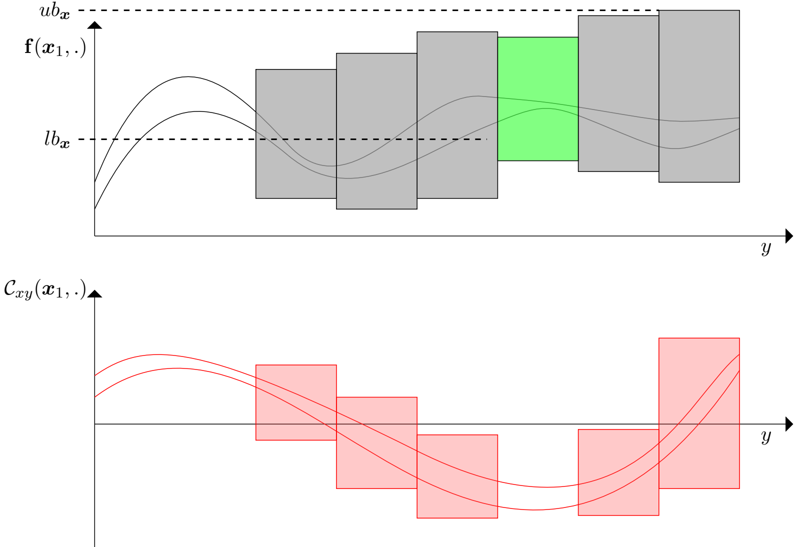

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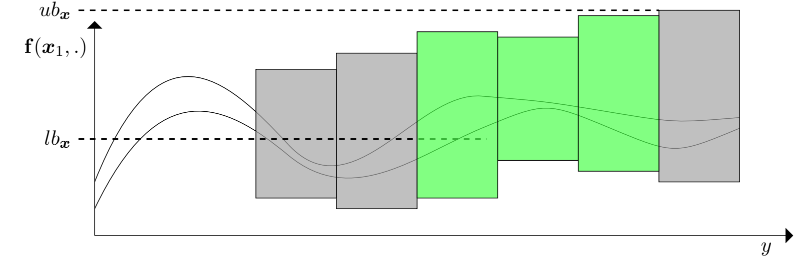

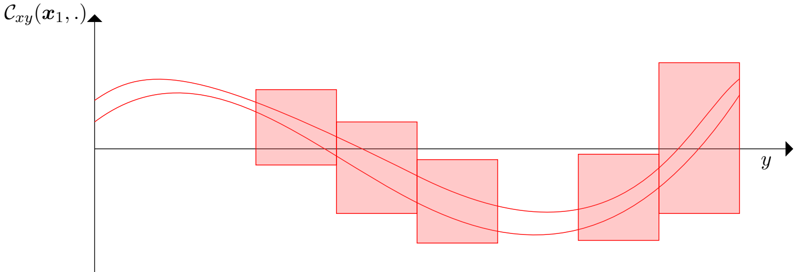

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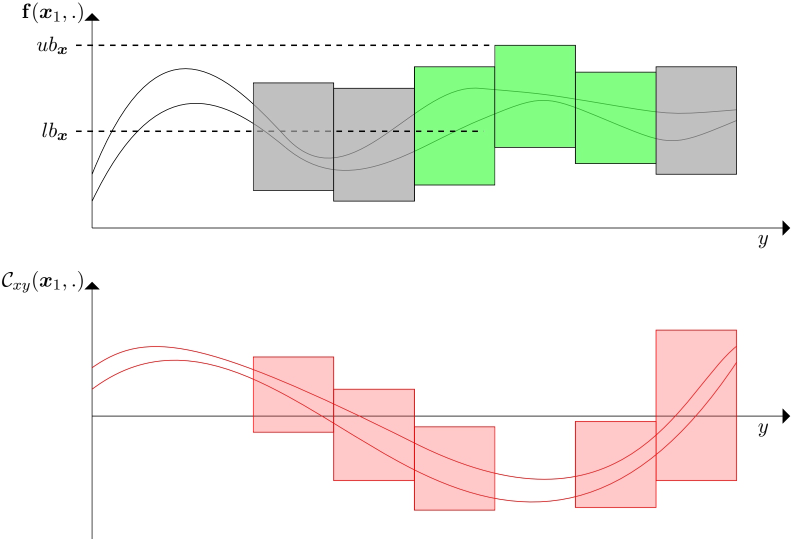

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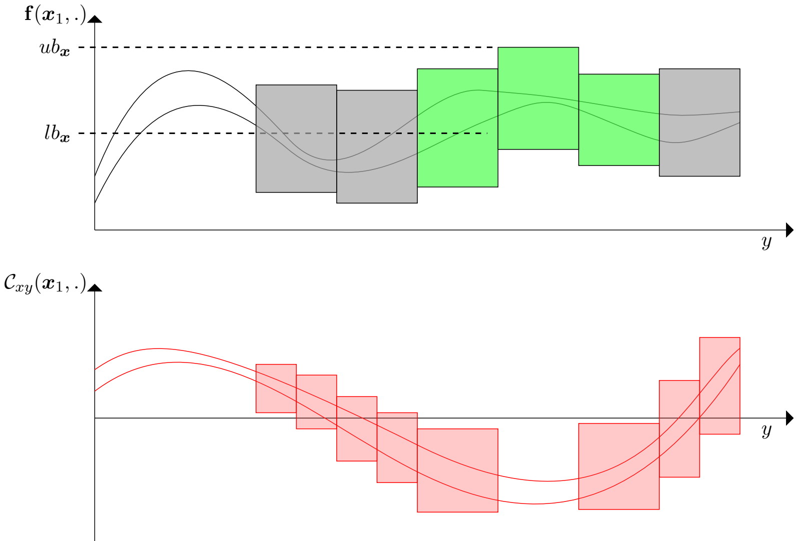

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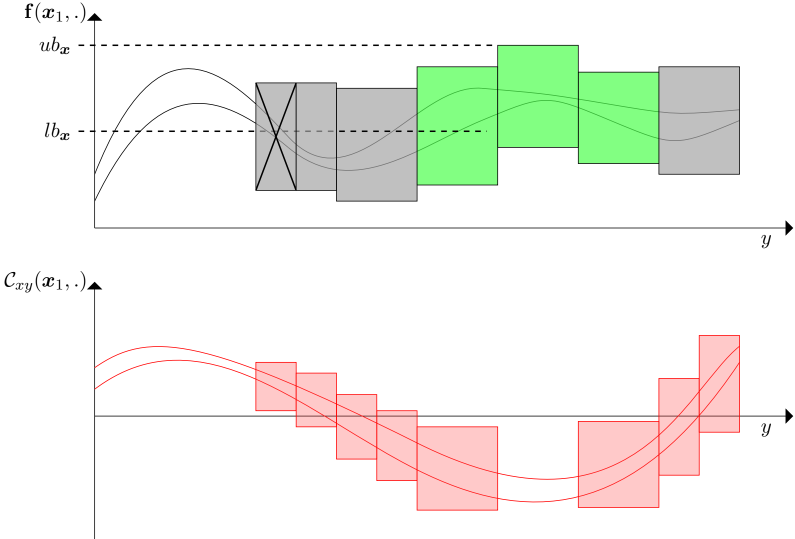

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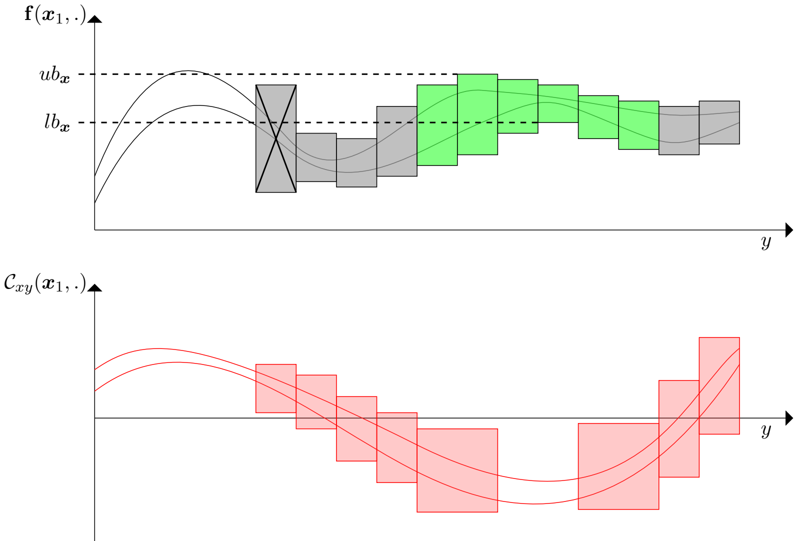

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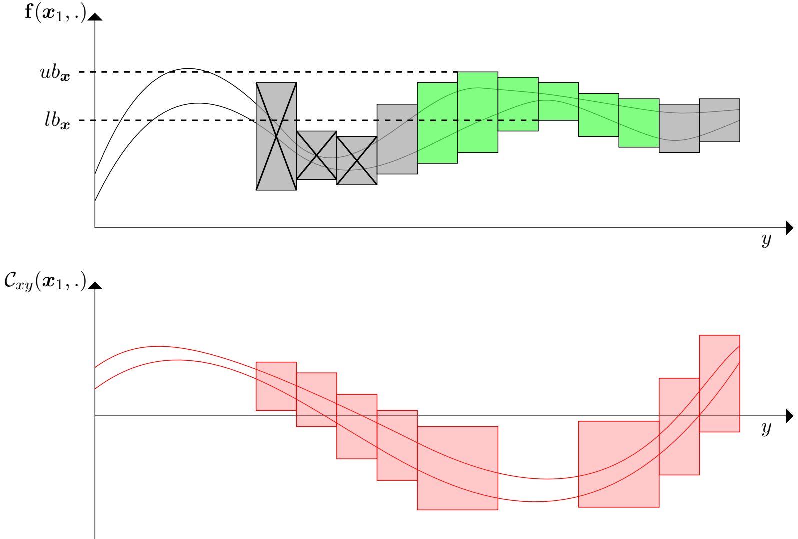

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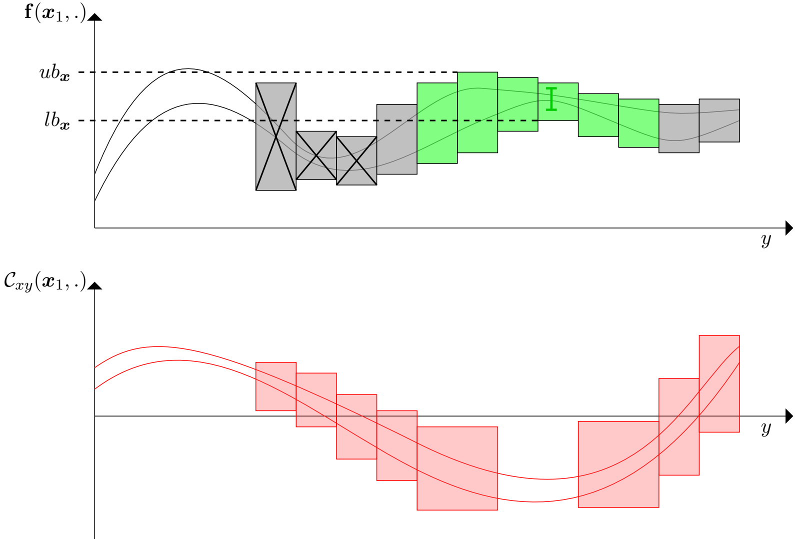

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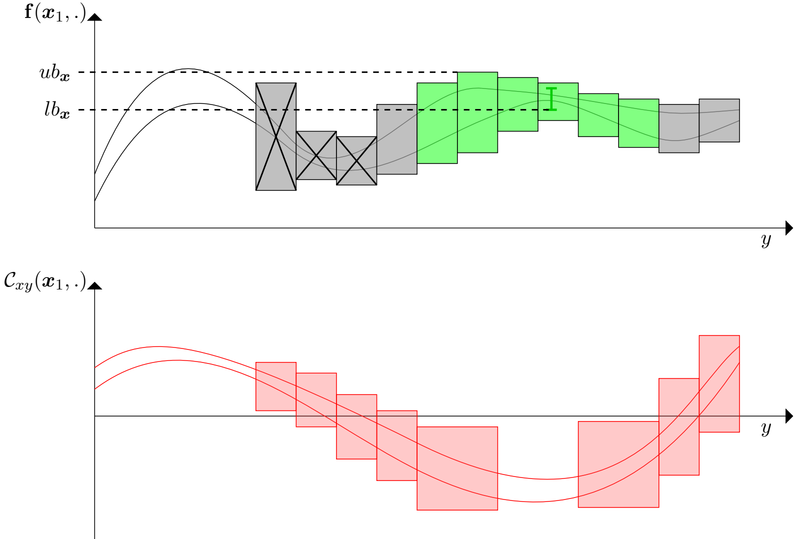

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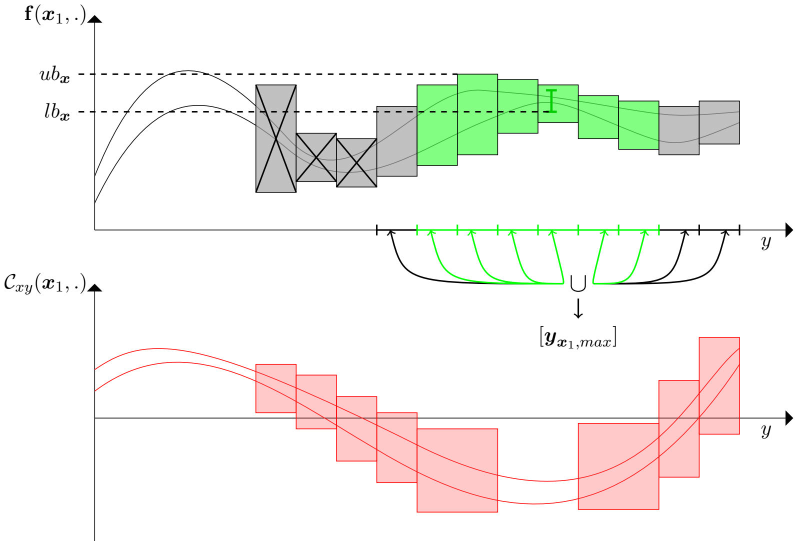

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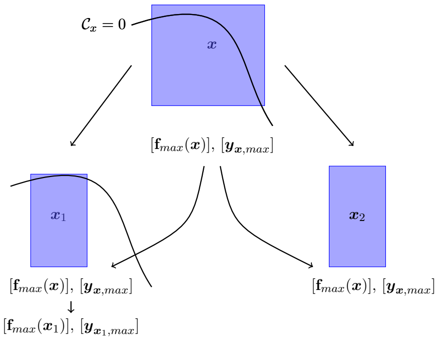

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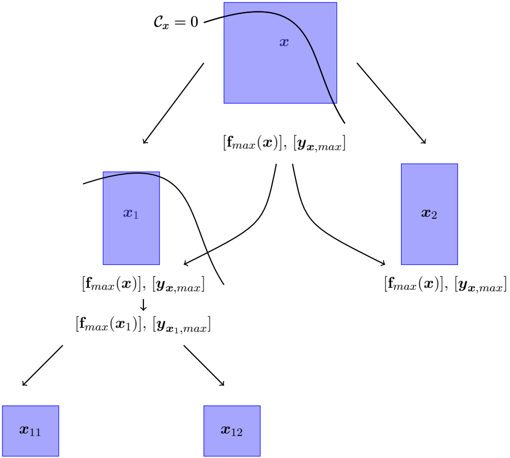

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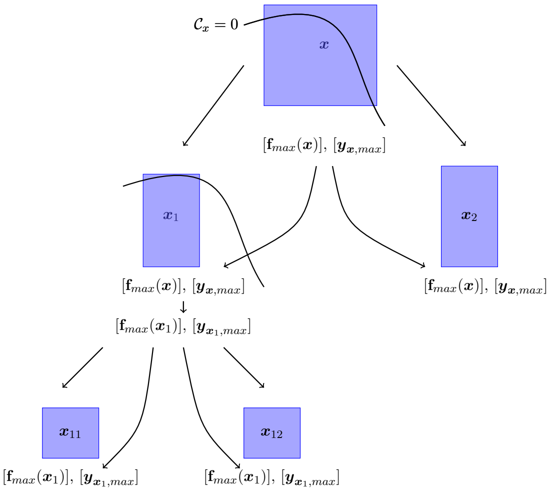


Plan

Examples

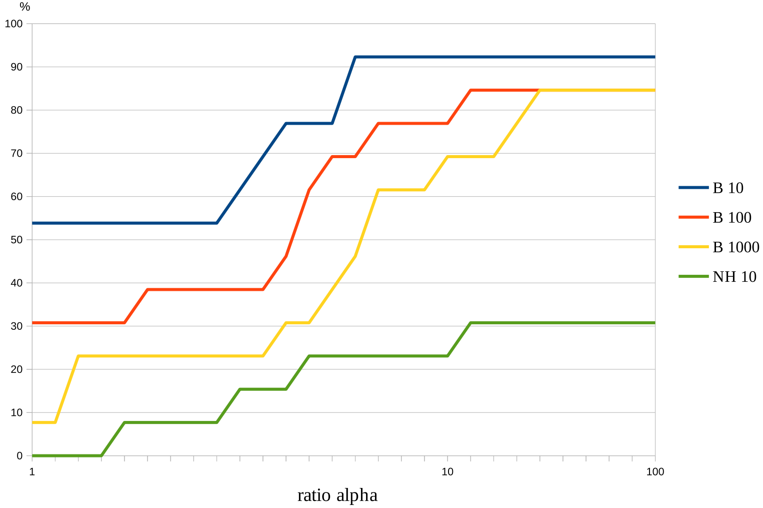
Problems	Obj. func.	x dim	y dim	$\mathcal{C}_x$	$\mathcal{C}_{xy}$
Article example[1]	other	2	2	no	no
Article example[3]	polynomial	1	1	no	yes
Article example[3]	trigonometric	1	1	no	yes
Control	rational	3	1	yes	no
Control	rational	4	1	yes	no
Control	rational	2	1	yes	no
Control	rational	4	1	yes	no
Control	rational	4	1	yes	no
Risk Management[2]	polynomial	2	2	no	no
Risk Management[2]	polynomial	2	2	no	no
Risk Management[2]	polynomial	2	2	no	no
Risk Management[2]	polynomial	2	3	no	no
Risk Management[2]	polynomial	3	3	no	no

Algorithm features

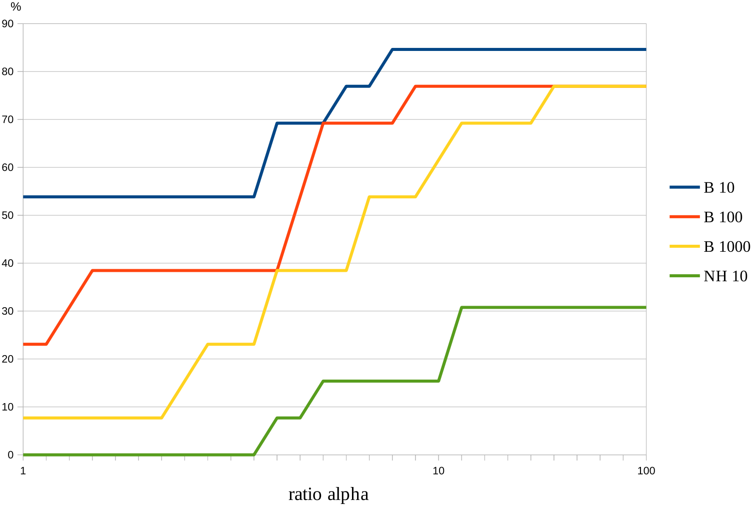
Algorithm is tested with four features:

- 10 bisections are performed in the maximization problem, inclusion properties used → B 10.
- 100 bisections are performed in the maximization problem, inclusion properties used → B 100.
- 1000 bisections are performed in the maximization problem, inclusion properties used → B 1000.
- 10 bisections are performed in the maximization problem, inheritance properties not used → NH 10.

Performance profile: cpu time



Performance profile: number of function evaluation



Plan

Conclusion

- Solver for non-convex problems (non-convex objective function and non-convex constraints).
- Taking advantage of Inclusion properties save computation time.
- Finding the best number of bisection is difficult.

Next steps:

- Test the algorithm on more examples.
- Improve convergence time (monotonicity tests, affine arithmetic, ...).
- How to find the number of bisection?

E. Carrizosa and F. Messine. A branch and bound method for global robust optimization. *Proc. 12th global optimization workshop (Málaga, Spain, September 2014)*, 2014.

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