A Global Optimization Approach to Structured Regulation Design under H_∞ Constraints

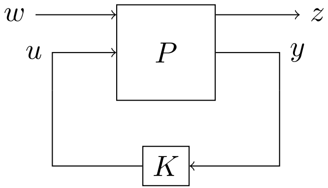
Dominique Monnet, Jordan Ninin, Benoît Clément
LAB-STICC, UMR 6285 / ENSTA-Bretagne

December 12, 2016

Plan

1. Introduction
2. Global optimization approach: Branch and Bound
3. H_∞ norm of MISO systems
4. Example
5. Conclusion

Structured H_∞ synthesis problem



P interconnected with K : $z = T_{w\to z} w$

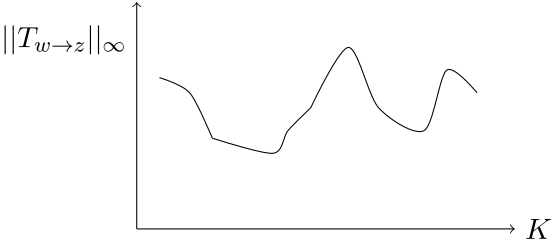
Structured H_∞ synthesis problem

$$\left\{ \begin{array}{l} \text{Find } K \in \mathbf{K}_s \text{ such that } ||T_{w\to z}||_\infty \leq \gamma \\ \text{and } K \text{ stabilizes } T_{w\to z} \end{array} \right.$$

or

$$\left\{ \begin{array}{l} \min\limits_{K \in \mathbf{K}_s} ||T_{w\to z}||_\infty \\ \text{subject to } K \text{ stabilizes } T_{w\to z} \end{array} \right.$$

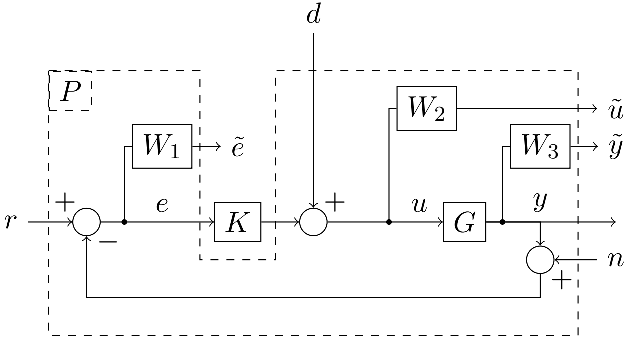
Problem solved?



- Convex approaches (LMI).
- Non convex approaches based on local optimization.
- Non convex approaches based on global optimization$\rightarrow$ only for SISO (polynomial formulation).

Global optimization for MIMO?

Almost possible



Constraints:

- $||T_{w\to\tilde{e}}||_\infty \leq 1$.
- $||T_{w\to\tilde{u}}||_\infty \leq 1$.
- $||T_{w\to\tilde{y}}||_\infty \leq 1$.

$||T_{w\to z}||_\infty$ is an over approximation:

$$\max(||T_{w\to\tilde{e}}||_\infty, ||T_{w\to\tilde{u}}||_\infty, ||T_{w\to\tilde{y}}||_\infty) \leq ||T_{w\to z}||_\infty$$

Almost possible

$K(k,s)$ is a parametric rational function.

PID example: $K(k,s) = k_p + \frac{k_i}{s} + k_d s$, $k = (k_p, k_i, k_d)$

Problem

$$\begin{cases} \text{Find } k \in \mathbf{k} \text{ such that } \|T_{w\to z_j}(k)\|_\infty \le \gamma,\ j \in \{1, ..., p\} \\ \text{and } K \text{ stabilizes } T_{w\to z} \end{cases}$$

or

$$\begin{cases} \min\limits_{k\in\mathbf{k}} \{\max\limits_{j} \|T_{w\to z_j}(k)\|_\infty\} \\ \text{subject to } K \text{ stabilizes } T_{w\to z} \end{cases}$$

The global optimization algorithm provides:

- A guaranteed enclosure $[l_b, u_b]$ of the minimum.
- A *good* controller K.
- And a certificate of unfeasibility if $l_b > \gamma$.

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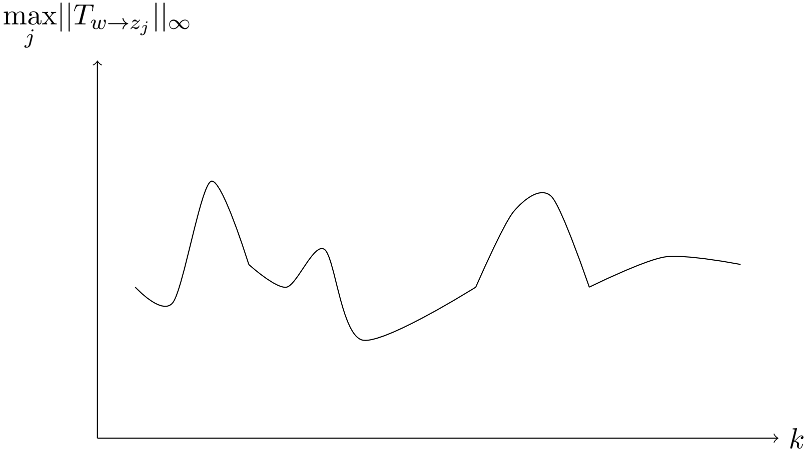
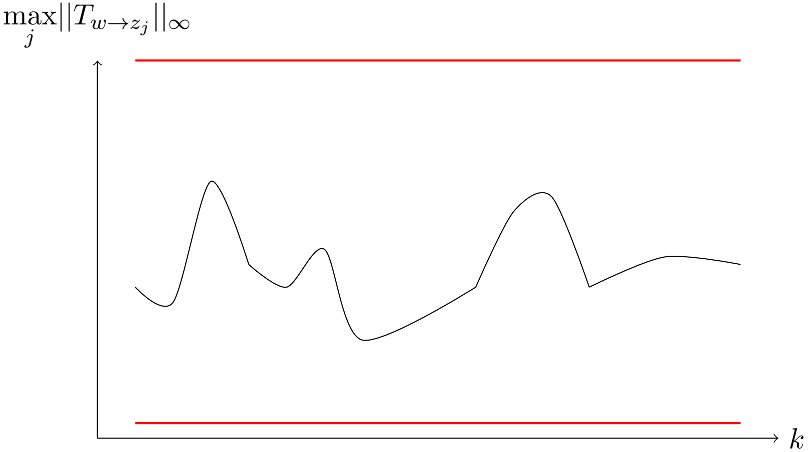
Branch and Bound Algorithm

Minimization problem

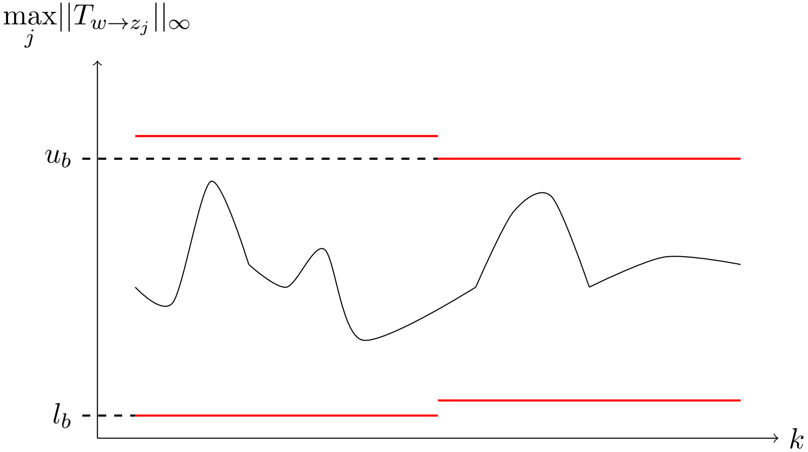
$$\min_{k\in\mathbf{k}}\left(\max_j ||T_{w\rightarrow z_j}(k)||_\infty\right)$$



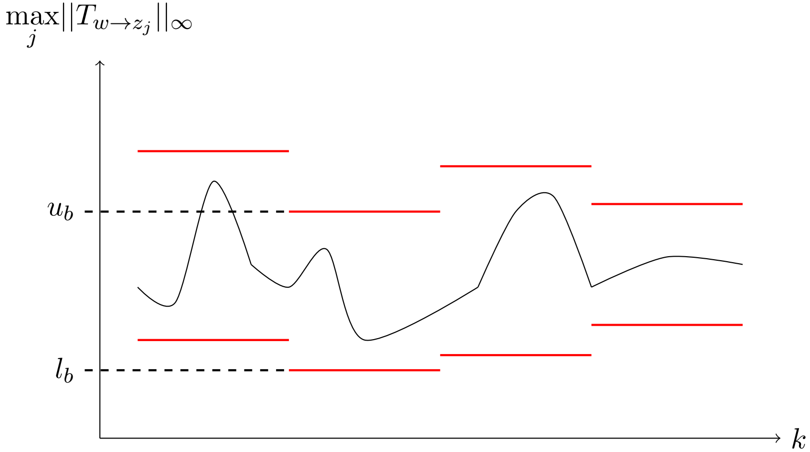
Branch and Bound Algorithm



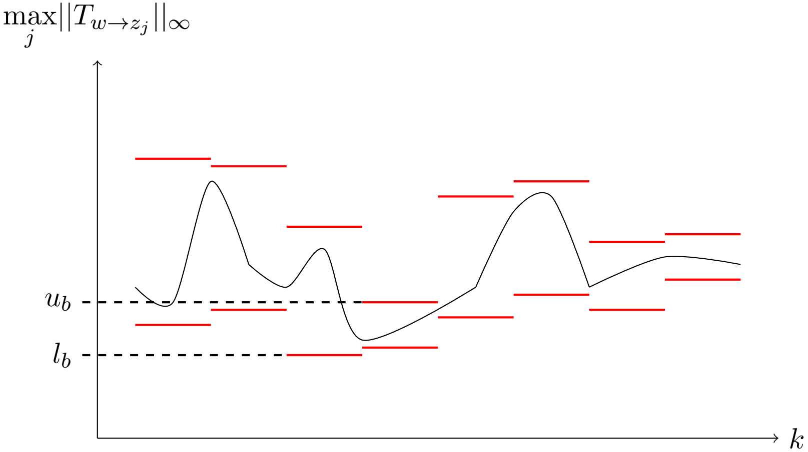
Branch and Bound Algorithm



Branch and Bound Algorithm



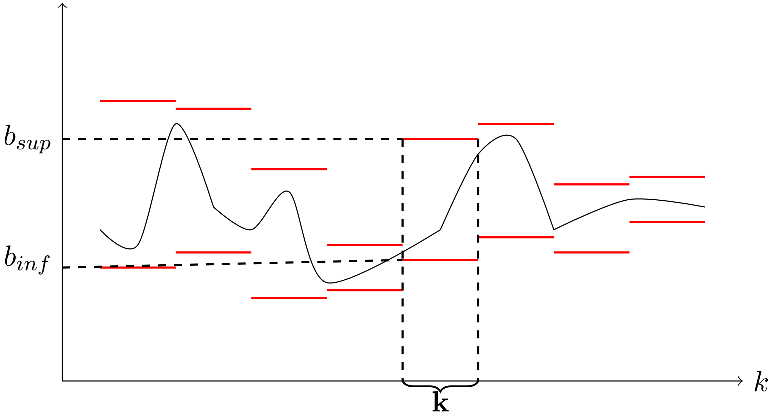
Branch and Bound Algorithm



Enclosure of infinity norm of MISO

Problem

Find b_{inf} and b_{sup} such that:
$b_{inf} \leq \max_j ||T_{w \to z_i}(k)||_\infty \leq b_{sup}, \forall k \in \mathbf{k}$



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Expression of $||.||_\infty$ of MISO systems

$T_{w\to z}(k,s)$ is a matrix of rational functions which maps w to z:

$$T_{w\to z}(k,s) = \begin{pmatrix} T_{w_1\to z_1}(k,s) & \dots & T_{w_n\to z_1}(k,s) \\ \vdots & & \vdots \\ T_{w_1\to z_p}(k,s) & \dots & T_{w_n\to z_p}(k,s) \end{pmatrix} = \begin{pmatrix} T_{w\to z_1}(k,s) \\ \vdots \\ T_{w\to z_p}(k,s) \end{pmatrix}$$

$$T_{w\to z_j}(k,s) = \begin{pmatrix} T_{w_1\to z_j}(k,s) & \dots & T_{w_n\to z_j}(k,s) \end{pmatrix}$$

Proposition

$$||T_{w\to z_j}(k)||_\infty = \sup_{\omega\geq 0}\sqrt{\sum_{l=1}^{n} |T_{w_l\to z_j}(k,i\omega)|^2}$$

Demonstration

$$||T_{w\to z_j}(k,s)||_\infty = \sup_{\omega\geq 0}\{\sigma_{max}(T_{w\to z_j}(k,i\omega))\}$$
$$= \sup_{\omega\geq 0}\sqrt{\lambda_{max}(T_{w\to z_j}(k,i\omega)T_{w\to z_j}(k,i\omega)^*)}$$
$$= \sup_{\omega\geq 0}\sqrt{\lambda_{max}\left(\left(T_{w_1\to z_j}(k,i\omega),\ldots T_{w_n\to z_j}(k,i\omega)\right)\begin{pmatrix}\overline{T_{w_1\to z_j}(k,i\omega)}\\ \vdots \\ \overline{T_{w_n\to z_j}(k,i\omega)}\end{pmatrix}\right)}$$
$$= \sup_{\omega\geq 0}\sqrt{\sum_{l=1}^{n}|T_{w_l\to z_j}(k,i\omega)|^2}$$

Maximization problem

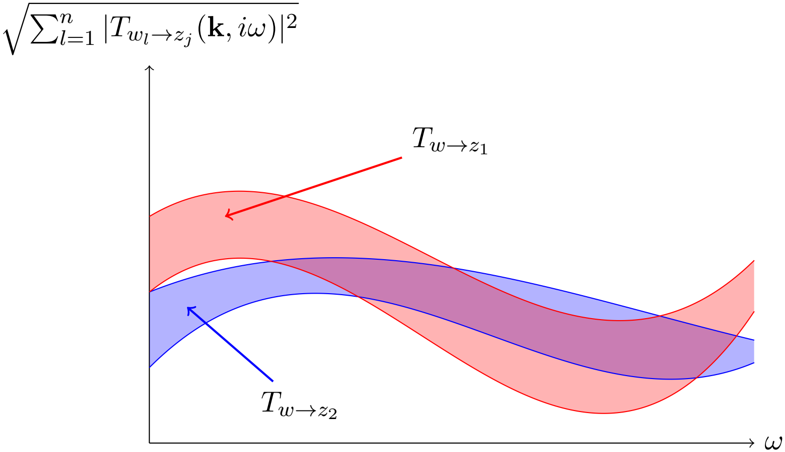
Problem

Find b_{inf} and b_{sup} such that:
$b_{inf} \leq \max_j ||T_{w \to z_i}(k)||_\infty \leq b_{sup}, \forall k \in \mathbf{k}$

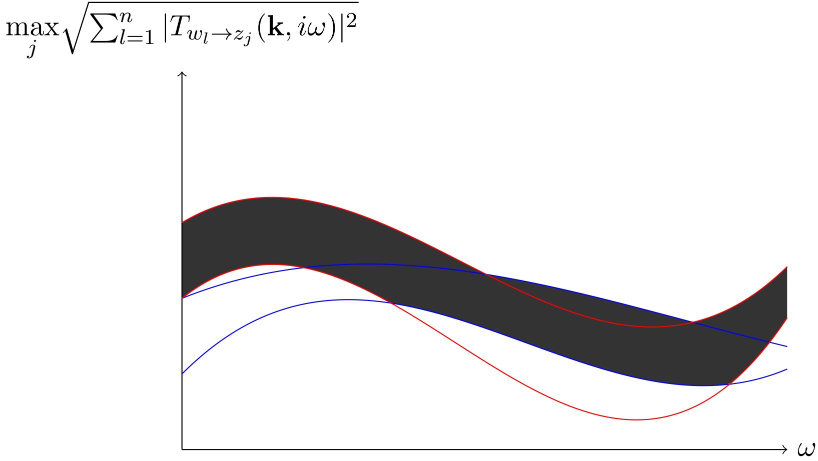
$$\max_j ||T_{w \to z_j}(\mathbf{k})||_\infty = \max_j \left\{ \sup_\omega \sqrt{\sum_{l=1}^{n} |T_{w_l \to z_j}(\mathbf{k}, i\omega)|^2} \right\}$$
$$= \sup_\omega \left\{ \max_j \sqrt{\sum_{l=1}^{n} |T_{w_l \to z_j}(\mathbf{k}, i\omega)|^2} \right\}$$

Use another branch and bound!
$\rightarrow$ Finite frequency range

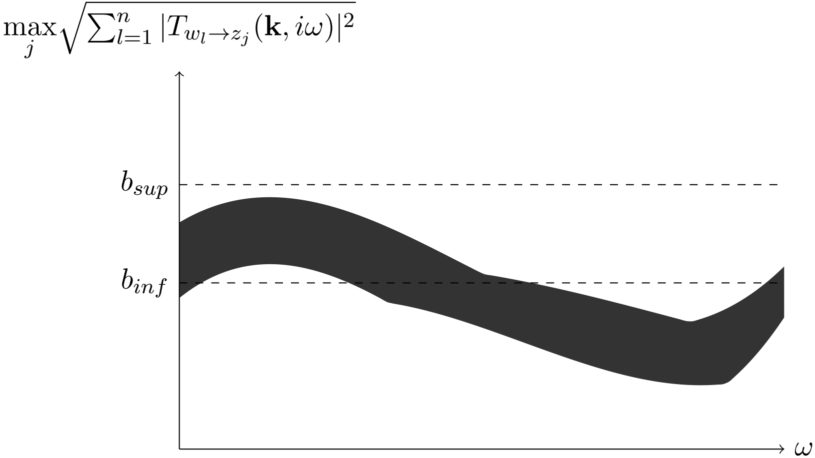
2^{nd} Branch and Bound



2^{nd} Branch and Bound



2^{nd} Branch and Bound



Min-Max formulation

The problem has a Min-Max formulation:

Min-Max Problem

$$\min_{k\in\mathbf{k}}\left(\sup_{\omega}\left\{\max_{j}\sqrt{\sum_{l=1}^{n}|T_{w_l\rightarrow z_j}(k,i\omega)|^2}\right\}\right)$$
s.t. K stabilizes the system: $P(k)\leq 0$

Complexity of a branch and bound: exponential.

$\rightarrow$ Branch and bound inside a branch and bound: Tricky problem.

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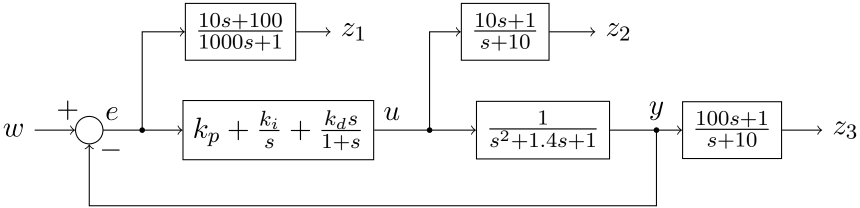
Example



$k_p \in [-10, 10]$, $k_i \in [-10, 10]$, $k_d \in [-10, 10]$

Method	Cpu (s)	$\|\|T_{w\to z}\|\|_\infty$	$\max_j(\|\|T_{w\to z_j}\|\|_\infty)$
H_∞ full	2	1.0258	1.0161
H_∞ struct	73 (500 rand start)	1.0411	1.0411
H_∞ hifoo	88 (30 rand start)	1.0349	1.0348
H_∞ systune	73 (2.1e6 rand start)	1.0986	0.9912
GO struct	80	1.0811	[0.9496,0.9947]

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Conclusion

- Analytic expression of $||.||_\infty$ of MISO system (square root of a rational function).
- Lower bound on the minimum $\implies$ we can prove that the CSP is not feasible.
- Another step towards structured robust synthesis? (yes).

$$G(\boldsymbol{\Delta}, s) \rightarrow \sup_{\omega,\Delta}\{\max_j \sqrt{\sum_{l=1}^{n} |T_{w_l \rightarrow z_j}(k, \Delta, i\omega)|^2}\}$$